Tutorial "General Relativity"

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Sheet No. 2 – Solutions

will be discussed on Nov/15/16

1. **Line Element**

Consider the two-dimensional line element given by

$$\mathrm{d}s^2 = x^2 \mathrm{d}x^2 + 2\mathrm{d}x\mathrm{d}y - \mathrm{d}y^2.$$

. Write down g_{ab}, g^{ab} and then raise and lower indices on $V_a = (1, -1)^T$ and $W^a = (0, 1)^T$.

Solution: The covariant metric components can be read off the expression for the line element as

$$(g_{ab}) = \hat{g} = \begin{pmatrix} x^2 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

The contravariant components are given by the inverse of this matrix, which is given by Kramer's rule, using $\det \hat{g} = -(1 + x^2)$,

$$(g^{ab}) = \hat{g}^{-1} = \frac{1}{1+x^2} \begin{pmatrix} 1 & 1 \\ 1 & -x^2 \end{pmatrix}. \tag{2}$$

The contravariant components of the vector $\boldsymbol{V}$, $V_a = (1, -1)$, are given by

$$(V^b) = (V_b g^{ba}) = (1, -1)\hat{g}^{-1} = (0, 1). \tag{3}$$

This implies

$$(W_a) = (g_{ab} W^b) = \hat{g} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4}$$

2. **Coordinate Transformations**

In a coordinate transformation, the components of the transformation matrix ${\Lambda^b}_a$ are formed by taking the partial derivative of one coordinate with respect to the other

$${\Lambda^b}_a = \frac{\partial x^b}{\partial x'^a},$$

whereas basis vectors transform as

$$\boldsymbol{e}'_a = {\Lambda^b}_a \boldsymbol{e}_b.$$

Plane polar coordinates are related to cartesian coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Describe the transformation matrix that maps cartesian coordinates to (holonomous) polar coordinates, and write down the polar-coordinate basis vectors in terms of the basis vectors of cartesian coordinates.

Solution: Contravariant vector components transform as the coordinate differentials,

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial x'^b}\mathrm{d}x'^b = \Lambda^a{}_b \mathrm{d}x'^b, \tag{5}$$

and vectors are invariant objects, i.e.,

$$\boldsymbol{V} = V^a \boldsymbol{e}_a = V'^b \boldsymbol{e}'_b = \Lambda^a{}_b V'^b \boldsymbol{e}_a \;\Rightarrow\; \boldsymbol{e}'_b = \Lambda^a{}_b \boldsymbol{e}_a. \tag{6}$$

That defines the transformations rules for contravariant (upper indices) and covariant (lower indices) objects like tensor components and basis vectors.

For the above example of polar coordinates (r, θ) we have

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{pmatrix} = \hat{\Lambda} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{pmatrix}. \tag{7}$$

From this we find

$$(\boldsymbol{e}_r, \boldsymbol{e}_\theta) = (\boldsymbol{e}_x, \boldsymbol{e}_y)\hat{\Lambda} = (\cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y, -r\sin\theta \boldsymbol{e}_x + r\cos\theta \boldsymbol{e}_y). \tag{8}$$

Note: Here we consider the socalled holonomous coordinates and basis vectors of the curvilinear coordinates, not the orthonormal basis vectors as usually used in three-dimensional vector calculus.

3. General Coordinate Transformations and Metric components

Under a coordinate transformation[1] $x^A = x^A(q^\mu)$, the Minkowski-metric components η_{AB} transform to new metric components $g_{\mu\nu}$ in such a way that proper distances are invariant. In other words, the line element $\mathrm{d}s^2 = \eta_{AB}\mathrm{d}x^A\mathrm{d}x^B$ is invariant, i.e., $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}q^\mu\mathrm{d}q^\nu$.

(a) Show, that this implies that $g_{\mu\nu}$ is related to η_{AB} by

$$g_{\mu\nu} = \frac{\partial x^A}{\partial q^\mu}\frac{\partial x^B}{\partial q^\nu}\eta_{AB}.$$

Solution: We have

$$\mathrm{d}s^2 = \eta_{AB}\mathrm{d}x^A\mathrm{d}x^B = \eta_{AB}\frac{\partial x^A}{\partial q^\mu}\frac{\partial x^B}{\partial q^\nu}\mathrm{d}q^\mu\mathrm{d}q^\nu =: g_{\mu\nu}\mathrm{d}q^\mu\mathrm{d}q^\nu. \tag{9}$$

Since this should hold true for all $\mathrm{d}q^\mu$, we must have

$$g_{\mu\nu} = \eta_{AB}\frac{\partial x^A}{\partial q^\mu}\frac{\partial x^B}{\partial q^\nu}. \tag{10}$$

(b) Show, that the inverse-metric components $g^{\mu\nu}$, i.e., $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$, are given by

$$g^{\mu\nu} = \eta^{AB}\frac{\partial q^\mu}{\partial x^A}\frac{\partial q^\nu}{\partial x^B}.$$

[1] Here we write capital roman letters to indicate components with respect to an inertial Minkowksi basis. As greek indices $A \in \{0, 1, 2, 3\}$, and the usual Einstein summation convention is used for these indices too.

Solution: We use (10) and the given equation to show that indeed $(g^{\mu\nu})$ is inverse to $(g_{\mu\nu})$:

$$
\begin{aligned}
g^{\mu\nu}g_{\nu\lambda} &= \eta^{AB}\frac{\partial q^\mu}{\partial x^A}\frac{\partial q^\nu}{\partial x^B}\eta_{CD}\frac{\partial x^C}{\partial q^\nu}\frac{\partial x^D}{\partial q^\lambda} \\
&= \left(\frac{\partial x^C}{\partial q^\nu}\frac{\partial q^\nu}{\partial x^B}\right)\frac{\partial x^D}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A}\eta^{AB}\eta_{CD} \\
&= \left(\frac{\partial x^C}{\partial x^B}\right)\frac{\partial x^D}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A}\eta^{AB}\eta_{CD} \\
&= \delta^C_B\frac{\partial x^D}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A}\eta^{AB}\eta_{CD} \\
&= \eta^{AB}\eta_{BD}\frac{\partial x^D}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A} = \delta^A_D\frac{\partial x^D}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A} \\
&= \frac{\partial x^A}{\partial q^\lambda}\frac{\partial q^\mu}{\partial x^A} = \frac{\partial q^\lambda}{\partial q^\mu} = \delta^\mu_\lambda.
\end{aligned}
\tag{11}
$$

4. Rotating frame in Special Relativity

A rotating frame can be described by

$$
\begin{aligned}
t &= t', \\
x &= x'\cos(\omega t') - y'\sin(\omega t'), \\
y &= x'\sin(\omega t)' + y'\cos(\omega t'), \\
z &= z'.
\end{aligned}
$$

The invariant line element reads $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$

(a) Calculate the metric components in the rotating frame.
Solution: We get (with $x^0 = x'^0 = ct = ct'$), using

$$
(\mathrm{d}x^\mu) = \mathrm{d}x'^\nu\frac{\partial x^\mu}{\partial x'^\nu} = \begin{pmatrix} \mathrm{d}x'^0 \\ \mathrm{d}x'\cos(\omega t') - \mathrm{d}t'\omega x'\sin(\omega t') - \mathrm{d}y'\sin(\omega t') - \mathrm{d}t'\omega y'\cos(\omega t') \\ -\mathrm{d}x'\sin(\omega t') + \mathrm{d}t'\omega x'\cos(\omega t') + \mathrm{d}y'\cos(\omega t') - \mathrm{d}t'\omega y'\sin(\omega t') \\ \mathrm{d}z' \end{pmatrix},
\tag{12}
$$

after some algebra

$$
\begin{aligned}
\mathrm{d}s^2 &= (\mathrm{d}x^0)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\
&= (\mathrm{d}x'^0)^2\left(1 - \frac{\omega^2}{c^2}(x'^2 + y'^2)\right) - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2 + 2\mathrm{d}x'^0\mathrm{d}x'\frac{\omega y'}{c} - 2\mathrm{d}x'^0\mathrm{d}y'\frac{\omega x'}{c}.
\end{aligned}
\tag{13}
$$

From this one reads off the covariant metric components in the new coordinates,

$$
(g'_{\mu\nu}) = \hat{g}' = \begin{pmatrix} 1 - \frac{\omega^2(x'^2+y'^2)}{c^2} & \frac{\omega y'}{c} & -\frac{\omega x'}{c} & 0 \\ \frac{\omega y'}{c} & -1 & 0 & 0 \\ -\frac{\omega x'}{c} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\tag{14}
$$

We also note the contravariant metric components, which are given by matrix inversion to

$$(g'^{\mu\nu}) = \hat{g}'^{-1} = \begin{pmatrix} 1 & \frac{\omega y'}{c} & \frac{-\omega x'}{c} & 0 \\ \frac{\omega y'}{c} & -1+\frac{\omega^2 y'^2}{c^2} & -\frac{\omega^2 x' y'}{c^2} & 0 \\ \frac{-\omega x'}{c} & -\frac{\omega^2 x' y'}{c^2} & -1+\frac{\omega^2 x'^2}{c^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{15}$$

(b) The affine connections (Christoffel symbols) for the primed coordinates are given as

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g'^{\rho\sigma} \left(\frac{\partial g'_{\nu\sigma}}{\partial x'^{\mu}} + \frac{\partial g'_{\mu\sigma}}{\partial x'^{\nu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} \right).$$

Calculate the non-vanishing affine connections.

(c) Derive the geodesic equation in a rotating frame. Use your results from (b) to derive the relativistic centrifugal- and the Coriolis force.

Hint: It is easier to first derive the equations of motion for the geodesic from the quadratic form of the Lagrangian,

$$L = \frac{1}{2} g'_{\mu\nu} \frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\lambda}, \tag{16}$$

i.e., using the Euler-Lagrange equations

$$g'^{\mu\nu} \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}'^{\nu}} - \frac{\partial L}{\partial x'^{\nu}} \right] = 0,$$

which then take directly the form of the geodesic equation (*proof that!*)

$$\frac{\mathrm{D}^2 x^{\mu}}{\mathrm{D}\lambda^2} := \frac{\mathrm{d}^2 x'^{\mu}}{\mathrm{d}\lambda^2} + \Gamma^{\mu}{}_{\alpha\beta} \frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^{\beta}}{\mathrm{d}\lambda} = 0.$$

From this it is easy to read off the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$.

Solution: Following the hint, we first prove the claimed connection between the Christoffel symbols and the Euler-Lagrange equations with the above given Lagrangian:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}'^{\nu}} = g'_{\nu\alpha} \ddot{x}'^{\alpha} + \partial_{\beta} g'_{\nu\alpha} \dot{x}'^{\alpha} \dot{x}'^{\beta} = g'_{\nu\alpha} \ddot{x}'^{\alpha} + \frac{1}{2} (\partial'_{\alpha} g'_{\beta\nu} + \partial'_{\beta} g'_{\alpha\nu}) \dot{x}'^{\alpha} \dot{x}'^{\beta} \tag{17}$$

$$\frac{\partial L}{\partial x'^{\nu}} = \frac{1}{2} \partial_{\nu} g'_{\alpha\beta} \dot{x}'^{\alpha} \dot{x}'^{\beta}. \tag{18}$$

Writing down the Euler-Lagrange equations and contracting with $g^{\mu\nu}$ finally leads to

$$\ddot{x}'^{\mu} + \frac{1}{2} g'^{\mu\nu} (\partial_{\alpha} g'_{\beta\nu} + \partial_{\beta} g'_{\alpha\nu} - \partial_{\nu} g'_{\alpha\beta}) = \ddot{x}'^{\mu} + \Gamma^{\mu}{}_{\alpha\beta} \dot{x}'^{\alpha} \dot{x}'^{\beta} = 0. \tag{19}$$

Now for the above example of a rotating reference frame, first we calculate

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}'^{\nu}} - \frac{\partial L}{\partial x'^{\nu}} \right) = \begin{pmatrix} (1+\omega^2\rho^2/c^2)\ddot{x}'^0 + \omega y' \ddot{x}'/c - \omega x' \ddot{y}'/c - 2\omega^2 \dot{x}'^0 (x'\dot{x}' + y'\dot{y}')/c^2 \\ \omega y' \ddot{x}'^0/c - \ddot{x}' + \omega^2 x' (\dot{x}'^0)^2/c^2 + 2\omega \dot{x}'^0 \dot{y}'/c \\ -\omega x' \ddot{x}'^0/c - \ddot{y}' - 2\omega \dot{x}'^0 \dot{x}'/c + \omega^2 y' (\dot{x}'^0)^2/c^2 \\ -\ddot{z}' \end{pmatrix} = 0. \tag{20}$$

Multiplying this covariant vector components with the contravariant metric $\hat{g}^{-1}$ we get

$$\begin{pmatrix} \ddot{x}'^0 \\ \ddot{x}' - \omega^2 x'(\dot{x}'^0)^2/c^2 - 2\omega\dot{x}'^0\dot{y}'/c \\ \ddot{y}' - \omega^2 y'(\dot{x}'^0)^2/c^2 + 2\omega\dot{x}'^0\dot{x}'/c \\ \ddot{z}' \end{pmatrix} = 0, \tag{21}$$

which is indeed in the form (19). We can immediately read off the non-vanishing Christoffel symbols,

$$\Gamma^1{}_{00} = -\frac{\omega^2 x'}{c^2}, \quad \Gamma^1{}_{02} = \Gamma^1{}_{20} = -\frac{\omega}{c}, \quad \Gamma^2{}_{00} = -\frac{\omega^2 y'}{c^2}, \quad \Gamma^2{}_{01} = \Gamma^2{}_{10} = \frac{\omega}{c}. \tag{22}$$

The geodesic equations are given by (21).

Since the Lagrangian (16) does not explicitly depend on the affine parameter λ the "Hamiltonian"

$$H = p'_\mu \dot{x}'^\mu - L, \quad p'_\mu = \frac{\partial L}{\partial \dot{x}'^\mu} = g'_{\mu\nu}\dot{x}'^\nu \tag{23}$$

is conserved. Now in our case $H = L$, and this implies that by choosing $H = c^2/2$ we define $\lambda = \tau$ to be the proper time. According to the first equation in (21) we have

$$x'^0 = ct = Ac\tau, \tag{24}$$

where we have choosen the origin of the coordinate time to coincide with the origin of proper time, and A is an integration constant to be determined. Then the spatial part of the equations of motion can be rewritten as

$$\ddot{\vec{x}}' + 2A\vec{\omega} \times \dot{\vec{x}}' + A^2\vec{\omega} \times (\vec{\omega} \times \vec{x}') = 0 \quad \text{with} \quad \vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}. \tag{25}$$

Multiplying with the invariant mass of the particle m and solving for $m\ddot{\vec{x}}'$ leads to the spatial components of the inertial Minkowski forces

$$m\ddot{\vec{x}}' = \vec{K}' = -2mA\vec{\omega} \times \dot{\vec{x}}' - mA^2\vec{\omega} \times (\vec{\omega} \times \vec{x}'). \tag{26}$$

It is clear that in this case the temporal component of the Minkowski force $K^0 = 0$.

(d) Solve the equations of motion with the choice $\lambda = \tau$ for the world-line parameter.

Hint: The only non-trivial equations are that for x' and y'. Here the task is tremendously simplified by introducing the complex auxilliary variable $\xi' = x' + \mathrm{i}y'$ and derive an equation of motion for it. Then the solution for x' and y' is given by $x' = \operatorname{Re}\xi'$ and $y' = \operatorname{Im}\xi'$.

Solution:

Written out in components the equations of motion (25)

$$\ddot{x}' - \omega^2 A^2 x' - 2\omega A\dot{y}' = 0, \quad \ddot{y}' - \omega^2 A^2 y' + 2\omega A\dot{x}' = 0, \quad \ddot{z}' = 0. \tag{27}$$

To solve the equations for x' and y' we introduce the complex variable

$$\xi' = x' + \mathrm{i}y'. \tag{28}$$

Then the first two equations (27) are obviously the real and imaginary parts of the equation

$$\ddot{\xi}' + 2\mathrm{i}\omega A\dot{\xi}' - \omega^2 A^2 \xi' = 0. \tag{29}$$

As a homogeneous linear differential equation of motion we try to solve it by making the ansatz

$$\xi'(\tau) = B \exp(-\mathrm{i}\Omega\tau). \tag{30}$$

Plugging this in (29) leads to the characteristic equation

$$(\Omega - A\omega)^2 = 0 \;\Rightarrow\; \Omega = A\omega. \tag{31}$$

Since we find only one solution for Ω, we need to find another independent solution of (29). To that end we insert the ansatz

$$\xi'(\tau) = B(\tau) \exp(-\mathrm{i}A\omega\tau), \tag{32}$$

leading to

$$\ddot{B}(\tau) = 0 \;\Rightarrow\; B(\tau) = B_1' + B_2'\tau, \quad B_1', B_2' \in \mathbb{C}. \tag{33}$$

Writing $B_1' = B_1 \exp(-\mathrm{i}\varphi_1)$, $B_2' = B_2 \exp(-\mathrm{i}\varphi_2)$ with $B_1, B_2 \geq 0$ the general solution of (29) is

$$\xi'(\tau) = B_1 \exp(-\mathrm{i}A\omega\tau + \mathrm{i}\varphi_1) + B_2\tau \exp(\mathrm{i}A\omega\tau + \mathrm{i}\varphi_2), \tag{34}$$

i.e.,

$$\begin{aligned} x'(\tau) &= \operatorname{Re}\xi'(\tau) = B_1 \cos(A\omega\tau + \varphi_1) + B_2\tau \cos(A\omega\tau + \varphi_2), \\ y'(\tau) &= \operatorname{Im}\xi'(\tau) = -B_1 \sin(A\omega\tau + \varphi_1) - B_2\tau \sin(A\omega\tau + \varphi_2). \end{aligned} \tag{35}$$

Of course, there are only six independent integration constants determined by the initial values $\vec{x}_0' = \vec{x}'(\tau = 0)$ and $\dot{\vec{x}}_0' = \dot{\vec{x}}'(\tau = 0)$. It is clear that in this case

$$A = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \text{const}, \tag{36}$$

and it is determined by

$$g_{\mu\nu}\dot{x}'^{\mu}\dot{x}'^{\nu} = c^2A^2 - B_2^2 - C_1^2 = c^2, \tag{37}$$

i.e.,

$$A = \sqrt{1 + \frac{B_2^2 + C_2^2}{c^2}}. \tag{38}$$