Tutorial "General Relativity"

Winter term 2016/2017

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Sheet No. 3 – Solutions

will be discussed on Nov/29/16

1. Tensor gymnastics

(a) Let $Q^{ab} = Q^{ba}$ be a symmetric tensor and $R^{ab} = -R^{ba}$ be an antisymmetric tensor. Show that

$$Q^{ab} R_{ab} = 0.$$

Solution: From the symmetry of Q^{ab} we have $Q^{ab} R_{ab} = Q^{ba} R_{ab}$, and from the antisymmetry of R^{ab} that is $Q^{ba} R_{ab} = -Q^{ba} R_{ba}$. Now we can rename the indices $b \to a$ and $a \to b$ over which is summed. So we have $Q^{ab} R_{ab} = -Q^{ab} R_{ab}$, from which necessarily follows $Q^{ab} R_{ab} = 0$.

(b) Let $Q^{ab} = Q^{ba}$ be a symmetric tensor and T_{ab} be an arbitrary tensor. Show that

$$T_{ab} Q^{ab} = \frac{1}{2} Q^{ab} \left(T_{ab} + T_{ba}\right).$$

Solution: From the symmetry of Q^{ab} we have $T_{ab} Q^{ab} = T_{ab} Q^{ba}$. Renaming again $a \leftrightarrow b$ gives $T_{ab} Q^{ba} = T_{ba} Q^{ab}$. Thus one has $2 T_{ab} Q^{ab} = Q^{ab} (T_{ab} + T_{ba})$, and that we had to prove.

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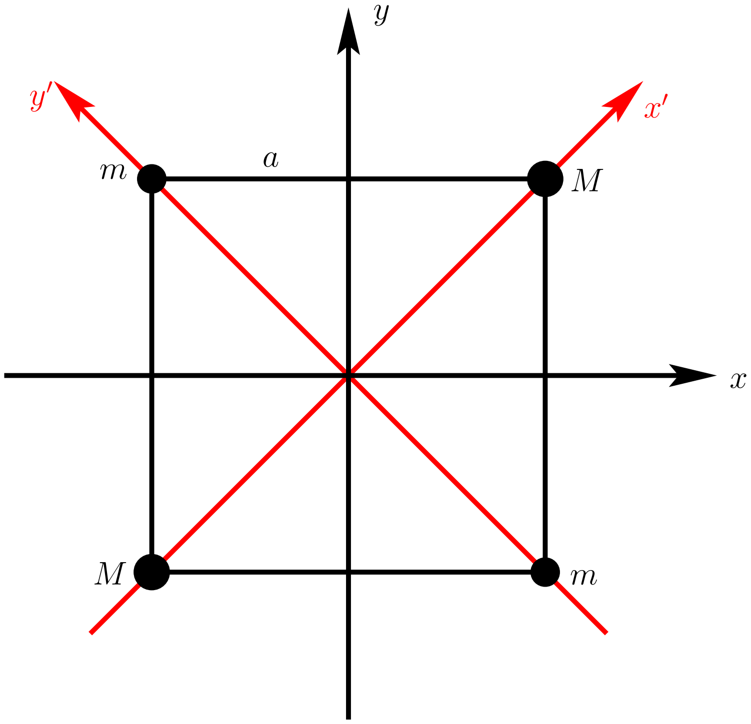
2. At the corners of a square of length $2a$ point masses m and M are located as shown in the figure in the xy plane ($z = 0$) of a Cartesian coordinate system.



(a) Calculate the components of the tensor of inertia,

$$\Theta^{ij} = \sum_k m_k(\vec{x}_k \cdot \vec{x}_k \delta^{ij} - x_k^i x_k^j),$$

with respect to the Cartesian coordinate sytem (x, y, z) and with respect to (x', y', z') (red), where $z = z'$. The sum over k runs over the four point masses.

Solution: In the original frame the position vectors of the masses can be read off the diagram as

$$\vec{x}_1 = (a, a, 0)^{\mathrm{T}}, \quad \vec{x}_2 = (-a, a, 0)^{\mathrm{T}}, \quad \vec{x}_3 = (-a, -a, 0)^{\mathrm{T}}, \quad \vec{x}_4 = (a, -a, 0), \tag{1}$$

and the masses are $m_1 = m_3 = M$ and $m_2 = m_4 = m$. Using the definition of the tensor of inertia given in the problem yields

$$\hat{\Theta} = (\Theta^{ij}) = 2a^2 \begin{pmatrix} m+M & -(M-m) & 0 \\ -(M-m) & m+M & 0 \\ 0 & 0 & 2(m+M) \end{pmatrix}. \tag{2}$$

In the primed frame we read off the coordinates of the mass points as

$$\vec{x}_1' = (a\sqrt{2}, 0, 0)^{\mathrm{T}}, \quad \vec{x}_2' = (0, a\sqrt{2}, 0)^{\mathrm{T}}, \quad \vec{x}_3' = (-a\sqrt{2}, 0, 0)^{\mathrm{T}}, \quad \vec{x}_4' = (0, -a\sqrt{2}, 0)^{\mathrm{T}}. \tag{3}$$

Using again the formula for the tensor components gives

$$\hat{\Theta}' = (\Theta'^{ij}) = 4a^2 \mathrm{diag}(m, M, m+M). \tag{4}$$

(b) Determine the rotation matrix ${D^k}_i$ that transforms the vector components of the position vectors according to $x'^k = {D^k}_i x^i$.

Solution: We have a counter-clock-wise rotation of the original frame by $\pi/4$ around the z axis. With $\cos \pi/4 = \sin \pi/4 = \sqrt{2}/2$ thus the rotation matrix reads

$$\hat{D} = ({D^i}_j) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

(c) Show through explicit calculation that the components of the tensor of inertia transform as the components of a second-rank tensor should, i.e., according to

$$\Theta'^{kl} = {D^k}_i {D^l}_j T^{ij}.$$

Solution: In matrix-product notation the formula claims

$$\hat{\Theta}' = \hat{D} \hat{\Theta} \hat{D}^{\mathrm{T}}, \tag{6}$$

and performing the matrix multiplications using (2) and (5) indeed leads to (4) as claimed.

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3. Geodesics[1]

A great circle of a sphere is the intersection of the sphere and a plane which passes through the center point of the sphere.

Show that geodesics on a sphere are great circles. Use

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

with $u^1 = \theta$, $u^2 = \phi$ and $R = \text{const}$.

[1] In this exercise we follow the lecture given by Prof. Greiner on Tuesday, November 08!

(a) Determine the metric and the affine connections given as

$$\Gamma^h{}_{ki} = \frac{1}{2} g^{hl} \left(\frac{\partial g_{lk}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^l} \right)$$

Solution The metric components read

$$\begin{aligned} g_{11} &= R^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = R^2 \sin^2\theta \\ g^{11} &= \frac{1}{R^2}, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{R^2 \sin^2\theta}, \end{aligned} \tag{7}$$

and thus the only non-vanishing Christoffel symbols are

$$\Gamma^1{}_{22} = -\sin\theta\cos\theta, \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta. \tag{8}$$

(b) Determine the geodesic equations for θ and ϕ, and show that these can be written as

$$\begin{aligned} \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}s} &= h = \text{const.} \\ \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \frac{h^2}{\sin^2\theta} &= \frac{h^2}{\sin^2\theta_0} = \text{const.} \end{aligned}$$

where h and $0 \lesssim \theta_0 \lesssim \frac{\pi}{2}$ are constants of integration.
Solution:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \Gamma^1{}_{ik} \frac{\mathrm{d}u^i}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s} = \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \Gamma^1{}_{22}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0, \tag{9}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \Gamma^1_{ik} \frac{\mathrm{d}u^i}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s} = \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \Gamma^2_{12} \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \Gamma^2_{21} \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{10}$$

Mutiplying Eq. (10) with $\sin^2\theta$ gives

$$\sin^2\theta \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\cos\theta\sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) = 0 \tag{11}$$

and thus the first equation

$$\sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}s} = h = \text{const.} \Rightarrow \frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{h}{\sin^2\theta} \tag{12}$$

Plugging this into (9) we obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \frac{h^2\cos\theta}{\sin^3\theta} = 0. \tag{13}$$

This can also be written as a total derivative by multiplying with $2\frac{\mathrm{d}\theta}{\mathrm{d}s}$

$$2\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - 2\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{h^2\cos\theta}{\sin^3\theta} = \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \frac{h^2}{\sin^2\theta}\right] = 0, \tag{14}$$

i.e.,

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \frac{h^2}{\sin^2\theta} = \frac{h^2}{\sin^2\theta_0} = \text{const.} \quad \text{with} \quad 0 \le \theta_0 \le \frac{\pi}{2}, \tag{15}$$

with the range for θ_0 follows from the fact that $\sin^2\theta_0$ is unique there.

(c) Use your results from (b) to determine $\mathrm{d}\phi/\mathrm{d}\theta$ and the function $\phi(\theta)$. Use

$$\int_{\theta_0}^{\theta} \mathrm{d}\theta' = \frac{1}{\sin\theta'\sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}} = \left[-\arctan\frac{\cos\theta'}{\sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}}\right]_{\theta_0}^{\theta}.$$

(d) Show, that

$$\cot\theta = \cot\theta_0 \cos(\phi - \phi_0)$$

Hint: Use the relations $\tan(x \pm \pi/2) = -\cot x$ and $1 + \cot^2 x = \frac{1}{\sin^2 x}$.
Solution: Applying the chain rule we find

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{d\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{d\phi}\frac{h}{\sin^2\theta}, \tag{16}$$

where we have used (12). Now we can use (16) in (15) to obtain

$$\begin{aligned} \frac{h^2}{\sin^4\theta}\left(\frac{\mathrm{d}\theta}{d\phi}\right)^2 + \frac{h^2}{\sin^2\theta} &= \frac{h^2}{\sin^2\theta_0}, \\ \Rightarrow \frac{\mathrm{d}\phi}{d\theta} = \pm\frac{1}{\sin\theta}\left(\frac{\sin^2\theta}{\sin^2\theta_0} - 1\right)^{-1/2} & . \end{aligned} \tag{17}$$

Due to (15) is $\theta \in \{\theta_0, \pi - \theta_0\}$ since (17) is only defined in this range. Separation of variables and integrating, using the integral given in the problem, leads to

$$\begin{aligned} \phi = \phi_0 \pm \int_{\theta_0}^{\theta} d\theta' \frac{1}{\sin\theta'\sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}} &= \phi_0 \pm \left[-\arctan\frac{\cos\theta'}{\sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}}\right]_{\theta_0}^{\theta} \\ &= \phi_0 \mp \arctan\frac{\cos\theta}{\sqrt{\frac{\sin^2\theta}{\sin^2\theta_0} - 1}} \pm \frac{\pi}{2} \end{aligned} \tag{18}$$

After some algebra this simplifies to

$$\begin{aligned} \tan\left(\phi - \phi_0 \mp \frac{\pi}{2}\right) = -\cot(\phi - \phi_0) &= \mp\frac{\cos\theta}{\sqrt{\frac{\sin^2\theta}{\sin^2\theta_0} - 1}} \\ &= \mp\frac{\cos\theta}{\sin\theta\sqrt{\frac{1}{\sin^2\theta_0} - \frac{1}{\sin^2\theta}}} = \mp\frac{\cot\theta}{\sqrt{\cot^2\theta_0 - \cot^2\theta}}, \end{aligned} \tag{19}$$

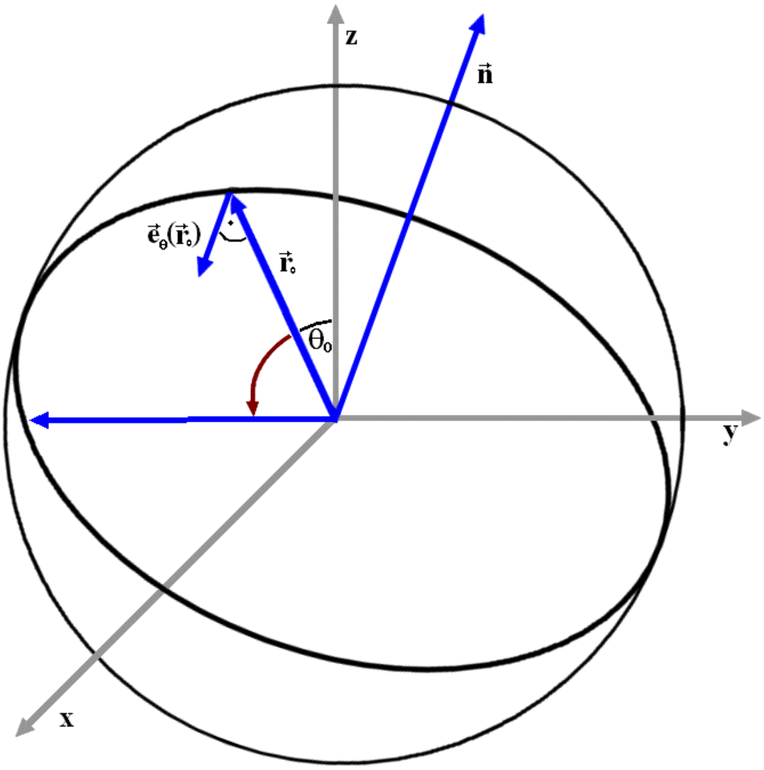
where we have used the given trigonometric relations. Finally we can resolve for $\cot\theta$:

$$\begin{aligned} &\cot^2(\phi - \phi_0)(\cot^2\theta_0 - \cot^2\theta) = \cot^2\theta \\ &\Rightarrow \cot^2(\phi - \phi_0)\cot^2\theta_0 = \cot^2\theta(1 + \cot^2(\theta - \theta_0)) \\ &\Rightarrow \cot\theta = \frac{\cot(\phi - \phi_0)\cot\theta_0}{\sqrt{1 + \cot^2(\phi - \phi_0)}} = \frac{\cot(\phi - \phi_0)\cot\theta_0}{\sqrt{\frac{1}{\sin^2(\phi - \phi_0)}}} = \cos(\phi - \phi_0)\cot\theta_0. \end{aligned} \tag{20}$$

To see that this indeed describes a great circle on the sphere we identify the point on the geodesic with coordinate (θ_0, ϕ_0) as its highest point. We get the great circle through this point,

$$\vec{r}_0 = R(\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0), \tag{21}$$

by rotating $\vec{r}_0$ around an axis $\vec{n}$ through the origin of the coordinate system that is perpendicular to $\vec{r}_0$. With respect to the axis $\vec{n}$ is this great circle through $\vec{r}_0$ the equator, and thus $\vec{n}$ is parallel to the unit vector $\vec{e}_\theta(\vec{r}_0)$ along a latitude. In the figure we show such a great circle with $\vec{r}_0$ as the point with the minimal angle $\theta = \theta_0$



The unit vector at this place is given by

$$\vec{n} = \vec{e}_\theta(\vec{r}_0) = \frac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}_{\theta_0,\phi_0} = \begin{pmatrix} \cos\theta_0\cos\phi_0 \\ \cos\theta_0\sin\phi_0 \\ -\sin\theta_0 \end{pmatrix}. \tag{22}$$

Only for a great circle the position vector along the cirlce $\vec{r}(\theta, \phi)$ is always perpendicular to this special unit vector in direction of growing θ. For smaller circles the scalar product won't vanish. Thus we find as a condition

$$\begin{aligned} \vec{e}_\theta(\theta_0, \phi_0) \cdot \vec{r}(\theta, \phi) &= R(\cos\theta_0\cos\phi_0\sin\theta\cos\phi + \cos\theta_0\sin\phi_0\sin\theta\sin\phi - \sin\theta_0\cos\theta) \\ &= R(\cos\theta_0\sin\theta(\cos\phi_0\cos\phi + \sin\phi_0\sin\phi) - \sin\theta_0\cos\theta) \\ &= R(\cos\theta_0\sin\theta\cos(\phi - \phi_0) - \sin\theta_0\cos\theta) \stackrel{!}{=} 0 \\ &\Rightarrow \cos\theta_0\sin\theta\cos(\phi - \phi_0) = \sin\theta_0\cos\theta \\ &\Rightarrow \cot\theta_0\cos(\phi - \phi_0) = \cot\theta, \end{aligned} \tag{23}$$

where in the 3rd line we have used $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$. Now according to (20) this condition is precisely the solution for the geodesic on the sphere.

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4. Additional problem (just for fun): Geodesics on the sphere simplified

That the great circles are the geodesics on the sphere can be much simpler derived by using that the geodesics equation can be derived by the variational principle with the Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

where $g_{ij} = g_{ij}(\vec{x})$ are the metric components and $\vec{x} = \vec{x}(\lambda)$ is a parametrization with a parameter λ that is automatically affine.

For the geodesics on the sphere use the Euclidean metric in $\mathbb{R}^3$ with components $g_{ij} = \delta_{ij}$ and implement the constraint $\vec{x}^2 = R^2 = \text{const}$ with a Lagrange parameter Λ leading to the Lagrangian

$$L = \frac{1}{2}\dot{\vec{x}}^2 - \frac{\Lambda}{2}(\vec{x}^2 - R^2).$$

(a) Derive the equations of motion from the variational principle (Euler-Lagrange equations).

Solution: First we calculate to canonical momenta:

$$\vec{p} = \frac{\partial L}{\partial \dot{x}} = \dot{\vec{x}}. \tag{24}$$

From this according to the Euler-Lagrange equations the geodesics are described by the differential equations

$$\dot{\vec{p}} = \ddot{\vec{x}} = \frac{\partial L}{\partial \vec{x}} = -\Lambda \vec{x}. \tag{25}$$

(b)

(c) To determine the Lagrange multiplier Λ, take the 2nd derivative of the constraint $\vec{x}^2 = R^2 =$ const with respect to λ, and then use the equations of motion from (a).

Hint: One can use the fact that λ is automatically an affine parameter, which can be normalized such that $\dot{\vec{x}}^2 = 1$.

Solution: Taking twice the derivative of the contraint leads to

$$\vec{x} \cdot \dot{\vec{x}} = 0, \quad \dot{\vec{x}}^2 + \vec{x} \cdot \ddot{\vec{x}} = 1 + \vec{x} \cdot \ddot{\vec{x}} = 0. \tag{26}$$

With the equations of motion (25) this leads to

$$1 - \Lambda \vec{x}^2 = 1 - \Lambda R^2 = 0 \;\Rightarrow\; \Lambda = \frac{1}{R^2}. \tag{27}$$

(d) Solve the equations of motion to show that it is a great circle on the sphere.

Solution: The equations of motion now read

$$\ddot{\vec{x}} = -\frac{1}{R^2}\vec{x}, \tag{28}$$

and the general solution is

$$\vec{x} = \vec{x}_1 \cos(\lambda/R) + \vec{x}_2 \sin(\lambda/R). \tag{29}$$

Now we have to work in the constraints. From the first equation (26) we get

$$\dot{\vec{x}} \cdot \vec{x} = \frac{1}{R}[-\vec{x}_1 \sin(\lambda/R) + \vec{x}_2 \cos(\lambda/R)] \cdot [\vec{x}_1 \cos(\lambda/R) + \vec{x}_2 \sin(\lambda/R)] = 0. \tag{30}$$

Now for $\lambda/R = 0$ this implies $\vec{x}_1 \cdot \vec{x}_2 = 0$, i.e., $\vec{x}_1$ and $\vec{x}_2$ are prependicular vectors. This implies that

$$\vec{x}^2 = \vec{x}_1^2 \cos^2(\lambda/R) + \vec{x}_2^2 \sin^2(\lambda/R) \stackrel{!}{=} R^2, \tag{31}$$

and now setting first $\lambda/R = 0$ gives $\vec{x}_1^2 = R^2$, and setting then $\lambda/R = \pi/2$ leads to $\vec{x}_2^2 = R^2$. Then (31) is indeed fulfilled for all λ. Since thus $\vec{x}_1$ and $\vec{x}_2$ are two perpendicular vector of length R, (29) indeed describes a circle with radius R around the origin, and this is of course a great circle on the sphere. This proves again that the geodesics on a sphere are great circles (or pieces of great circles).