Tutorial “General Relativity”

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Sheet No. 2

will be discussed on Nov/15/17

1. Line Element

Consider the two-dimensional line element given by

$$\mathrm{d}s^2 = x^2\mathrm{d}x^2 + 2\mathrm{d}x\mathrm{d}y - \mathrm{d}y^2$$

. Write down g_{ab}, g^{ab} and then raise and lower indices on $V_a = (1, -1)^T$ and $W^a = (0, 1)^T$.

2. Coordinate Transformations

In a coordinate transformation, the components of the transformation matrix ${\Lambda^b}_a$ are formed by taking the partial derivative of one coordinate with respect to the other

$${\Lambda^b}_a = \frac{\partial x^b}{\partial x'^a},$$

whereas basis vectors transform as

$$\boldsymbol{e}'_a = {\Lambda^b}_a \boldsymbol{e}_b$$

Plane polar coordinates are related to cartesian coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta$$

Describe the transformation matrix that maps cartesian coordinates to (holonomous) polar coordinates, and write down the polar-coordinate basis vectors in terms of the basis vectors of cartesian coordinates.

3. General Coordinate Transformations and Metric components

Under a coordinate transformation[1] $x^A = x^A(q^\mu)$, the Minkowski-metric components η_{AB} transform to new metric components $g_{\mu\nu}$ in such a way that proper distances are invariant. In other words, the line element $\mathrm{d}s^2 = \eta_{AB}\mathrm{d}x^A\mathrm{d}x^B$ is invariant, i.e., $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}q^\mu\mathrm{d}q^\nu$.

(a) Show, that this implies that $g_{\mu\nu}$ is related to η_{AB} by

$$g_{\mu\nu} = \frac{\partial x^A}{\partial q^\mu}\frac{\partial x^B}{\partial q^\nu}\eta_{AB}.$$

(b) Show, that the inverse metric $g^{\mu\nu}$, i.e., $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$ is given by

$$g^{\mu\nu} = \eta^{AB}\frac{\partial q^\mu}{\partial x^A}\frac{\partial q^\nu}{\partial x^B}.$$

[1] Here we write capital roman letters to indicate components with respect to an inertial Minkowksi basis. As greek indices $A \in \{0, 1, 2, 3\}$, and the usual Einstein summation convention is used for these indices too.

4. **Rotating frame in Special Relativity**

A rotating frame can be described by

$$\begin{aligned} t &= t', \\ x &= x'\cos(\omega t') - y'\sin(\omega t'), \\ y &= x'\sin(\omega t') + y'\cos(\omega t'), \\ z &= z'. \end{aligned}$$

The invariant line element reads $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$

(a) Calculate the line element and read off the metric components in the rotating frame.

(b) The affine connections (Christoffel symbols) for the primed coordinates are given as

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x'^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x'^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x'^{\sigma}} \right)$$

Calculate the non-vanishing affine connections.

(c) Derive the geodesic equation in a rotating frame. Use your results from (b) to derive the relativistic centrifugal- and the Coriolis force.

Hint: It is easier to first derive the equations of motion for the geodesic from the quadratic form of the Lagrangian,

$$L = \frac{1}{2} g_{\mu\nu} \frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\lambda},$$

i.e., using the Euler-Lagrange equations[2]

$$g^{\mu\nu} \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}'^{\nu}} - \frac{\partial L}{\partial x'^{\nu}} \right] = 0,$$

which then take directly the form of the geodesic equation (*proof that!*)

$$\frac{\mathrm{D}^2 x'^{\mu}}{\mathrm{D}\lambda^2} := \frac{\mathrm{d}^2 x'^{\mu}}{\mathrm{d}\lambda^2} + \Gamma^{\mu}{}_{\alpha\beta} \frac{\mathrm{d}x'^{\alpha}}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^{\beta}}{\mathrm{d}\lambda} = 0.$$

From this it is easy to read off the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$.

Since the Lagrangian is not explicitly dependent on the "world-line parameter" λ,

$$H = p'_{\mu}\dot{x}'^{\mu} - L = L = \text{const.} \quad \text{with} \quad p'_{\mu} = \frac{\partial L}{\partial \dot{x}'^{\mu}}.$$

This implies that one can choose $\lambda = \tau$, i.e., the proper time of the particle, as the world-line parameter by normalizing it such that

$$g_{\mu\nu} \frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}x'^{\nu}}{\mathrm{d}\lambda} = 2L = c^2.$$

(d) Solve the equations of motion with the choice $\lambda = \tau$ for the world-line parameter.

Hint: The only non-trivial equations are that for x' and y'. Here the task is tremendously simplified by introducing the complex auxilliary variable $\xi' = x' + \mathrm{i}y'$ and derive an equation of motion for it. Then the solution for x' and y' is given by $x' = \operatorname{Re}\xi'$ and $y' = \operatorname{Im}\xi'$.

[2] Here $\dot{x}'^{\nu} = \mathrm{d}x'^{\nu}/\mathrm{d}\lambda$.