Tutorial "General Relativity"

Winter term 2016/2017

Lecturer: Prof. Dr. C. Greiner
Tutor: Hendrik van Hees

Sheet No. 3

will be discussed on Nov/29/16

1. Tensor gymnastics

(a) Let $Q^{ab} = Q^{ba}$ be a symmetric tensor and $R^{ab} = -R^{ba}$ be an antisymmetric tensor. Show that

$$Q^{ab} R_{ab} = 0.$$

(b) Let $Q^{ab} = Q^{ba}$ be a symmetric tensor and T_{ab} be an arbitrary tensor. Show that

$$T_{ab} Q^{ab} = \frac{1}{2} Q^{ab} \left(T_{ab} + T_{ba} \right).$$

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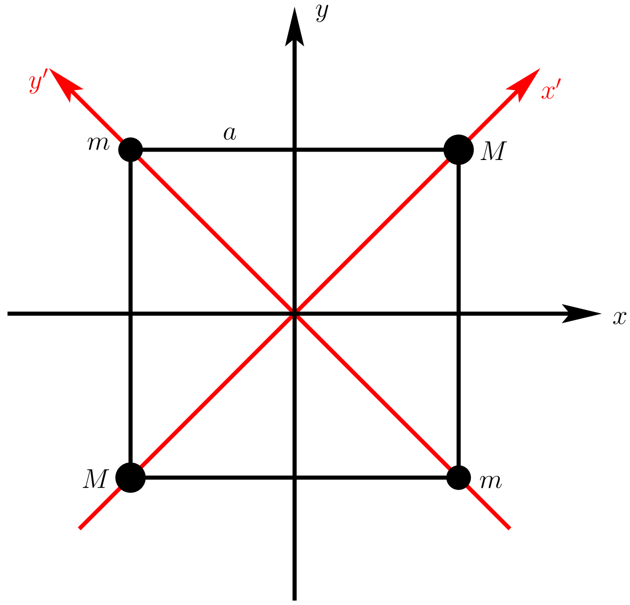
2. At the corners of a square of edge length $2a$ point masses m and M are located as shown in the figure in the xy plane ($z = 0$) of a Cartesian coordinate system.



(a) Calculate the components of the tensor of inertia,

$$\Theta^{ij} = \sum_k m_k (\vec{x}_k \cdot \vec{x}_k \delta^{ij} - x_k^i x_k^j),$$

with respect to the Cartesian coordinate sytem (x, y, z) and with respect to (x', y', z') (red), where $z = z'$. The sum over k runs over the four point masses.

(b) Determine the rotation matrix ${D^k}_i$ that transforms the vector components of the position vectors according to $x'^k = {D^k}_i x^i$.

(c) Show through explicit calculation that the components of the tensor of inertia transform as the components of a second-rank tensor should, i.e., according to

$$\Theta'^{kl} = {D^k}_i {D^l}_j T^{ij}.$$

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3. **Geodesics**[1]

A great circle of a sphere is the intersection of the sphere and a plane which passes through the center point of the sphere.

Show that geodesics on a sphere are great circles. Use

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

with $u^1 = \theta$, $u^2 = \phi$ and $R = \text{const.}$

(a) Determine the metric and the affine connections given as

$$\Gamma^h{}_{ki} = \frac{1}{2} g^{hl} \left(\frac{\partial g_{lk}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^l} \right)$$

(b) Determine the geodesic equations for θ and ϕ, and show that these can be written as

$$\sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}s} = h = \text{const.}$$
$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + \frac{h^2}{\sin^2\theta} = \frac{h^2}{\sin^2\theta_0} = \text{const.}$$

where h and $0 \lesssim \theta_0 \lesssim \frac{\pi}{2}$ are constants of integration.

(c) Use your results from (b) to determine $\mathrm{d}\phi/\mathrm{d}\theta$ and the function $\phi(\theta)$. Use:

$$\int_{\theta_0}^{\theta} \mathrm{d}\theta' = \frac{1}{\sin\theta' \sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}} = \left[-\arctan \frac{\cos\theta'}{\sqrt{\frac{\sin^2\theta'}{\sin^2\theta_0} - 1}} \right]_{\theta_0}^{\theta}$$

(d) Show, that

$$\cot\theta = \cot\theta_0 \cos(\phi - \phi_0)$$

Hint: Use the relations $\tan(x \pm \pi/2) = -\cot x$ and $1 + \cot^2 x = \frac{1}{\sin^2 x}$.

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4. **Additional problem (just for fun): Geodesics on the sphere simplified**

That the great circles are the geodesics on the sphere can be much simpler derived by using that the geodesics equation can be derived by the variational principle with the Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

where $g_{ij} = g_{ij}(\vec{x})$ are the metric components and $\vec{x} = \vec{x}(\lambda)$ is a parametrization with a parameter λ that is automatically affine.

For the geodesics on the sphere use the Euclidean metric in $\mathbb{R}^3$ with components $g_{ij} = \delta_{ij}$ and implement the constraint $\vec{x}^2 = R^2 = \text{const}$ with a Lagrange parameter Λ leading to the Lagrangian

$$L = \frac{1}{2}\dot{\vec{x}}^2 - \frac{\Lambda}{2}(\vec{x}^2 - R^2).$$

(a) Derive the equations of motion from the variational principle (Euler-Lagrange equations).

(b) To determine the Lagrange multiplier Λ, take the 2nd derivative of the constraint $\vec{x}^2 = R^2 = \text{const}$ with respect to λ, and then use the equations of motion from (a).

Hint: One can use the fact that λ is automatically an affine parameter, which can be normalized such that $\dot{\vec{x}}^2 = 1$.

(c) Solve the equations of motion to show that it is indeed a great circle on the sphere.

[1] In this exercise we follow the lecture given by Prof. Greiner on Tuesday, November 08!