Höhere Quantenmechanik – Advanced Quantum Mechanics

1. The formalism of quantum mechanics

2. Many-body quantum mechanics

3. Relativistic formulations

Literature:

in general:

[Bjorken] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics, Mc Graw-Hill

[Brown] L.S. Brown, Quantum Field Theory, Cambridge University Press

[Cohen] C. Cohen-Tannoudji, B. Diu, F. Laloe, Quantum Mechanics Vol. 1+2, Wiley

[Peskin] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory

Perseus

[Sakurai] J.J. Sakurai, Modern Quantum Mechanics, Addison-Wesley

[Scherz] U. Scherz, Quantenmechanik, Teubner

what I used for specific parts of my lecture:

basics of QM: [Brown] chapter 1.1, [Sakurai] chapters 1, 2

many-particle systems: [Brown] chapters 2.1, 2.2

Klein-Gordon equation: [Brown] chapters 3.1, 3.2

Dirac equation: [Bjorken] chapters 1-4, 6

scattering theory: [Brown] chapters 3.3, 3.4

Hartree-Fock method: [Scherz] Kapitel 7

1 The formalism of quantum mechanics

1.1. Recapitulating some facts of linear algebra

a) Motivation: In quantum mechanics a physical state, e.g. a particle or an N-particle system, is mathematically described by a state in a Hilbert space. This is very similar to a vector in a vector space.

b) Vectors and their representation

Consider a vector $\vec{v}$ in 2- or 3- or, more general, N-dimensional space. If a coordinate system S is specified, one can represent $\vec{v}$ by an N-component object (also called „vector"

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \qquad (1)$$

Typically one uses coordinate systems built from an orthonormal basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_N$ with

$$\vec{e}_i \cdot \vec{e}_j = \left\{ \begin{matrix} 1 & \text{for } i=j \\ 0 & \text{else} \end{matrix} \right\} = \delta_{ij}, \text{ Kronecker delta } (i,j=1,\ldots,$$

orthogonality

normalization

The connection between the (abstract) vector $\vec{v}$ and the representation (1) is given by

$$\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i \qquad (2)$$

The vectors $\vec{e}_1, \ldots, \vec{e}_N$ form a basis, i.e. are in particular <u>complete</u>, if <u>every</u> vector $\vec{v}$ can be represented like in (2).

Examples: consider in three-dimensional space the unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$ along the coordinate axes x, y, z.

α) $\vec{e}_x, \vec{e}_y$ form an orthonormal system, but it is incomplete since $\vec{e}_z$ cannot be represented by a linear combination of $\vec{e}_x$ and $\vec{e}_y$

β) $\vec{e}_x, \vec{e}_y, 5\vec{e}_z$ are an orthogonal basis, but not normalised

γ) $\vec{e}_x, \frac{1}{\sqrt{2}}(\vec{e}_x+\vec{e}_y), \vec{e}_z$ are a normalized basis, but not orthogonal

So far we have tacitly assumed that $v_i \in \mathbb{R}$ for all $i=1,..,N$ — at least these are the vector spaces which we can imagine for $N=2$ or $N=3$

To put our considerations slightly closer to quantum mechanics, let us consider a vector space $\mathbb{C}^N$ instead of $\mathbb{R}^N$, i.e.

$v_i \in \mathbb{C}$ for $i=1,..,N$
complex numbers

Still we keep

$$\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i$$

c) Scalar product

So far we have not specified what $\vec{e}_i \cdot \vec{e}_j$ actually means. We shall also change the notation to $\langle \vec{e}_i | \vec{e}_j \rangle$ which fits better to complex vector space.

For two vectors $\vec{v}, \vec{w}$ the scalar product is calculated in the following way: Use the representations $(v_1,..,v_N)$, $(w_1,..,w_N)$ and calculate complex conjugation

$$\sum_{i=1}^{N} v_i^{*} w_i =: \langle \vec{v} | \vec{w} \rangle$$

The scalar product satisfies $\langle \vec{v} | \vec{w} \rangle^{*} = \langle \vec{w} | \vec{v} \rangle$,

$$\langle \vec{u} | \alpha \vec{v} + \beta \vec{w} \rangle = \alpha \langle \vec{u} | \vec{v} \rangle + \beta \langle \vec{u} | \vec{w} \rangle$$

and

$$\langle \alpha \vec{u} + \beta \vec{v} | \vec{w} \rangle = \alpha^{*} \langle \vec{u} | \vec{w} \rangle + \beta^{*} \langle \vec{v} | \vec{w} \rangle$$

!

for arbitrary complex numbers $\alpha, \beta \in \mathbb{C}$ and arbitrary vectors $\vec{u}, \vec{v}, \vec{w}$

With the help of the scalar product one can reexpress the v_i's:

$$\langle \vec{e}_j | \vec{v} \rangle = \langle \vec{e}_j | \sum_{i=1}^{N} v_i \vec{e}_i \rangle = \sum_i v_i \underbrace{\langle \vec{e}_j | \vec{e}_i \rangle}_{= \delta_{ij}} = v_j$$

only contribution from sum, if $i = j$

$$\Rightarrow \quad v_i = \langle \vec{e}_i | \vec{v} \rangle \quad \text{for all } i = 1, \dots, N$$

$$\Rightarrow \quad \vec{v} = \sum_{i=1}^{N} \vec{e}_i \langle \vec{e}_i | \vec{v} \rangle \qquad (3)$$

d) Dirac's bra-ket notation

In quantum mechanics one is in the end interested in matrix elements. As we will see below, this corresponds to scalar products. As long as we keep $\vec{v}$ and $\vec{w}$ arbitrary, the most general thing, which we want to know, is

$$\langle \vec{w} | \vec{v} \rangle = \sum_{i=1}^{N} \langle \vec{w} | \vec{e}_i \rangle \langle \vec{e}_i | \vec{v} \rangle$$

Now we abstract from the scalar product $\langle \vec{w} | \vec{v} \rangle$ the two objects

$\langle \vec{w} |$ bra (or bra-vector)

and $| \vec{v} \rangle$ ket (or ket-vector)

which form a bra-ket = bracket = scalar product $\langle \vec{w} | \vec{v} \rangle$.

The scalar product

$$\langle \vec{w} | \vec{v} \rangle = \sum_{i=1}^{N} \langle \vec{w} | \vec{e}_i \rangle \langle \vec{e}_i | \vec{v} \rangle$$

holds for every vector $\vec{w}$. Therefore one can drop $\langle \vec{w} |$ without loss of information:

$$| \vec{v} \rangle = \sum_{i=1}^{N} | \vec{e}_i \rangle \langle \vec{e}_i | \vec{v} \rangle$$

which is just the content of equation (3).
In other words $| \vec{v} \rangle$ just denotes the vector $\vec{v}$.
On the other hand, also $\langle \vec{w} |$ is a vector. It "lives" in the dual vector space (see books on linear algebra).

e) linear, hermitean and unitary operators

(Linear) operators σ transform $\vec{v}$ into another vector $\vec{u}$

$$\vec{u} = \sigma \vec{v} \quad \text{or} \quad | \vec{u} \rangle = | \sigma \vec{v} \rangle =: \sigma | \vec{v} \rangle$$

σ can be represented by an $N \times N$ matrix with entries $\sigma_{ij} \in \mathbb{C}$, $i, j = 1, .., N$.

The components of the new vector $\vec{u}$ are

$$u_i = \sum_{j=1}^{N} \sigma_{ij} v_j$$

scalar product:

$$\langle \vec{w} | \vec{u} \rangle = \langle \vec{w} | \sigma \vec{v} \rangle = \langle \vec{w} | \sigma | \vec{v} \rangle$$

on the other hand:

$$\langle \vec{w} | \vec{u} \rangle = \sum_i w_i^* u_i = \sum_{ij} w_i^* \sigma_{ij} v_j$$

$$= \sum_{ij} (w_i \sigma_{ij}^*)^* v_j = \sum_{ij} (w_i (\sigma^T)_{ji}^*)^* v_j$$

$$= \sum_{ij} ((\sigma^+)_{ji} w_i)^* v_j = \langle \sigma^+ \vec{w} | \vec{v} \rangle$$

$$\Rightarrow \langle \vec{w} | \sigma \vec{v} \rangle = \langle \sigma^+ \vec{w} | \vec{v} \rangle$$

where we have introduced the adjoint operator σ^+ and the transposed operator σ^T.

σ^T is represented by an $N \times N$ matrix with entries σ_{ji}, i.e. by exchanging rows with columns in σ_{ij}

σ^+ is represented by an $N \times N$ matrix with entries σ_{ji}^*

Note: $(|\vec{v}\rangle\langle\vec{w}|)^+ = |\vec{w}\rangle\langle\vec{v}|$ for all vectors $\vec{v}, \vec{w}$ (without proof here)

An operator σ is called self-adjoint or hermitian (we do not make a distinction here, see text books on linear algebra), if $\sigma = \sigma^+$, i.e. $\sigma_{ij} = \sigma_{ji}^*$ for all $i, j = 1, \dots, N$

For hermitian operators one finds

$$\langle \vec{w} | \sigma \vec{v} \rangle = \langle \vec{w} | \sigma | \vec{v} \rangle = \langle \sigma \vec{w} | \vec{v} \rangle$$

for all vectors $\vec{v}, \vec{w}$

An operator $\mathcal{U}$ is called unitary, if

$$\mathcal{U}^+ = \mathcal{U}^{-1}$$

i.e. if $\mathcal{U}^+ \mathcal{U} \vec{v} = \vec{v}$ for all $\vec{v}$

One can also express it in terms of scalar products:
$\mathcal{U}$ is unitary, if

$$\langle \mathcal{U}\vec{w} | \mathcal{U}\vec{v} \rangle = \langle \vec{w} | \mathcal{U}^+ \mathcal{U} \vec{v} \rangle \overset{!}{=} \langle \vec{w} | \vec{v} \rangle$$

for all $\vec{v}, \vec{w}$

f) properties of the basis $\vec{e}_1, \dots, \vec{e}_N$

α) orthogonality: $\boxed{\langle \vec{e}_i | \vec{e}_j \rangle = 0}$ for $i \neq j$

β) normalization: $\boxed{\langle \vec{e}_i | \vec{e}_i \rangle = 1}$ for all $i = 1, \dots, N$

γ) completeness: from $|\vec{v}\rangle = \sum_i |\vec{e}_i\rangle \langle \vec{e}_i | \vec{v} \rangle$

$$\Rightarrow \boxed{\sum_i |\vec{e}_i\rangle \langle \vec{e}_i| = \mathbb{1}}$$

unity operator

these properties lead to

$$v_i = \langle \vec{e}_i | \vec{v} \rangle \quad \text{(see page 3)}$$

and

$$\sum_{ij} w_i^* \sigma_{ij} v_j = \langle \vec{w} | \sigma | \vec{v} \rangle = \langle \vec{w} | \Big(\sum_i |\vec{e}_i\rangle\langle\vec{e}_i| \Big) \sigma \Big(\sum_j |\vec{e}_j\rangle\langle\vec{e}_j| \Big) | \vec{v} \rangle$$

$$= \sum_{ij} \underbrace{\langle \vec{w} | \vec{e}_i \rangle}_{w_i^*} \langle \vec{e}_i | \sigma | \vec{e}_j \rangle \underbrace{\langle \vec{e}_j | \vec{v} \rangle}_{v_j} \quad \text{for all } \vec{v}, \vec{w}$$

$$\Rightarrow \sigma_{ij} = \langle \vec{e}_i | \sigma | \vec{e}_j \rangle$$

g) changing the representation

A vector in space does not change, if one uses a coordinate system S' different from the original one S, but the representation changes. One takes a column vector $\begin{pmatrix} v_1' \\ \vdots \\ v_N' \end{pmatrix}$ or

$$|\vec{v}\rangle = \sum_{i=1}^{N} v_i' |\vec{e}_i'\rangle$$

where the vectors $\vec{e}_1', \vec{e}_2', \dots \vec{e}_N'$ from S'.

We demand again an orthonormal complete set:

$$\langle \vec{e}_i' | \vec{e}_j' \rangle = \delta_{ij}$$

$$\sum_{i=1} |\vec{e}_i'\rangle\langle \vec{e}_i'| = \mathbb{1}$$

One finds $v_i' = \langle \vec{e}_i' | \vec{v} \rangle$.

Relation between v_i's and v_is:

$$v_i' = \langle \vec{e}_i' | \vec{v} \rangle = \sum_j \langle \vec{e}_i' | \vec{e}_j \rangle \underbrace{\langle \vec{e}_j | \vec{v} \rangle}_{v_j} = \sum_j \langle \vec{e}_i' | \vec{e}_j \rangle v_j$$

$\Rightarrow$ a linear operator $\mathcal{U}$ represented by the matrix elements

$$\mathcal{U}_{ij} = \langle \vec{e}_i' | \vec{e}_j \rangle \quad \text{(in the coordinate system } S) \qquad (4)$$

connects the representations v_i and v_i'

the scalar product is unchanged by changing the representation:

$$\sum_i w_i'^* v_i' = \sum_i \langle \vec{w} | \vec{e}_i' \rangle \langle \vec{e}_i' | \vec{v} \rangle = \langle \vec{w} | \underbrace{\left(\sum_i |\vec{e}_i'\rangle\langle \vec{e}_i'| \right)}_{= \mathbb{1}} | \vec{v} \rangle$$

$$= \langle \vec{w} | \vec{v} \rangle = \sum_i w_i^* v_i$$

Since the scalar product is unchanged and on the other hand, unitary operators leave the scalar product untouched, one may suspect that $\mathcal{U}$ is unitary

$\leadsto$ indeed

$$\langle \vec{w} | \mathcal{U}^\dagger \mathcal{U} | \vec{v} \rangle = \langle \mathcal{U}\vec{w} | \mathcal{U}\vec{v} \rangle = \sum_{ijk} (\mathcal{U}_{ij} w_j)^* \mathcal{U}_{ik} v_k$$

$$= \sum_{ijk} \left(\langle \vec{e}_i{}' | \vec{e}_j \rangle \right)^* \langle \vec{w} | \vec{e}_j \rangle \langle \vec{e}_i{}' | \vec{e}_k \rangle \langle \vec{e}_k | \vec{v} \rangle$$

$$= \sum_{ijk} \langle \vec{w} | \vec{e}_j \rangle \langle \vec{e}_j | \vec{e}_i{}' \rangle \langle \vec{e}_i{}' | \vec{e}_k \rangle \langle \vec{e}_k | \vec{v} \rangle$$

$$= \langle \vec{w} | \vec{v} \rangle \quad \text{for all } \vec{v}, \vec{w}$$

$$\Rightarrow \quad \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$$

note: $\mathcal{U} = \sum_i | \vec{e}_i \rangle \langle \vec{e}_i{}' |$

proof: calculate representation of $\mathcal{U}$ in coordinate system S:

$$\mathcal{U}_{kl} = \langle \vec{e}_k | \mathcal{U} | \vec{e}_l \rangle = \sum_i \underbrace{\langle \vec{e}_k | \vec{e}_i \rangle}_{\delta_{ki}} \langle \vec{e}_i{}' | \vec{e}_l \rangle$$

$$= \langle \vec{e}_k{}' | \vec{e}_l \rangle \quad \text{(agrees with (4) at page 7)}$$

h) exponentials of operators

for an operator $\hat{O}$ one can define (if it converges)

$$e^{\hat{O}} := \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \hat{O}^n$$

if one defines $\hat{O}^0 := \mathbb{1}$ for every operator $\hat{O}$, one can even write

$$e^{\hat{O}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{O}^n$$

In general: $e^A e^B \neq e^{A+B}$ (see Baker-Campbell-Hausdorff formula)

But if A and B commute, i.e. $[A,B] := AB - BA = 0$

$\Rightarrow e^A e^B = e^{A+B}$ (proof: the same as for numbers)

The inverse of $e^{\hat{O}}$ is $e^{-\hat{O}}$, i.e. $(e^{\hat{O}})^{-1} = e^{-\hat{O}}$

proof: $e^{-\hat{O}} e^{\hat{O}} \underset{[-\hat{O},\hat{O}]=0}{=} e^{-\hat{O}+\hat{O}} = e^{0} = \mathbb{1}$

If $\hat{O}$ is a hermitean operator, then $e^{i\hat{O}}$ is unitary

proof: $(e^{i\hat{O}})^+ = \sum_{n=0}^{\infty} \frac{1}{n!} [i^n \hat{O}^n]^+ = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \hat{O}^n$

$= e^{-i\hat{O}} = (e^{i\hat{O}})^{-1}$

i) unitary transformations

Let $\mathcal{U}$ be a unitary operator and consider the transformation

$$\vec{v} \to \mathcal{U}\vec{v} \quad \text{for all } \vec{v}$$

$$\sigma \to \mathcal{U}\,\sigma\,\mathcal{U}^+ \quad \text{for all operators } \sigma$$

This transformation leaves the scalar product $\langle \vec{w} | \sigma | \vec{v} \rangle$ unchanged.

The eigen values of $\mathcal{U}\sigma\mathcal{U}^+$ and σ are identical:

$$\sigma\,|\vec{v}_i\rangle = \sigma_i\,|\vec{v}_i\rangle$$

$$\Rightarrow \mathcal{U}\sigma\,\mathcal{U}^+\underbrace{\mathcal{U}\,|\vec{v}_i\rangle}_{\text{new eigen vectors}} = \sigma_i\,\mathcal{U}\,|\vec{v}_i\rangle$$

If the commutator between two operators σ_1, σ_2 is just a number α, then this fact is not changed by a unitary transformation:

$$[\mathcal{U}\sigma_1\mathcal{U}^+, \mathcal{U}\,\sigma_2\,\mathcal{U}^+] = \mathcal{U}\,\sigma_1\,\sigma_2\,\mathcal{U}^+ - \mathcal{U}\,\sigma_2\,\sigma_1\,\mathcal{U}^+$$
$$= \mathcal{U}\,[\sigma_1, \sigma_2]\,\mathcal{U}^+ = \alpha\,\mathcal{U}\,\mathcal{U}^+ = \alpha$$

(In quantum mechanics this is e.g. relevant for the commutator of position and momentum)

j) projection operators

suppose one has a 3-dimensional vector (v_x, v_y, v_z) and wants to know its projection onto the x-y plane

$$P_{xy}\vec{v} = (v_x, v_y, 0) = \vec{v} - \underbrace{(\vec{v}\cdot\vec{e}_z)}_{v_z}\underbrace{\vec{e}_z}_{(0,0,1)} = \sum_{i=x,y}(\vec{v}\cdot\vec{e}_i)\,\vec{e}_i$$

more general:

$$P = \sum_{i=1}^{M} |\vec{e}_i\rangle\langle\vec{e}_i| \quad \text{is a projector with } M < N$$

the property which defines a projector is

$$Q^2 = Q$$

indeed

$$P^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} |\vec{e}_i\rangle\underbrace{\langle\vec{e}_i|\vec{e}_j\rangle}_{\delta_{ij}}\langle\vec{e}_j| = \sum_{i=1}^{M}|\vec{e}_i\rangle\langle\vec{e}_i| = P$$

If Q is a projector, also $\mathbb{1} - Q$ is a projector:

$$(\mathbb{1}-Q)^2 = \mathbb{1} - 2Q + \underbrace{Q^2}_{=Q} = \mathbb{1} - Q$$

Q and $1-Q$ project on orthogonal subspaces:

$$Q(1-Q) = Q - Q = 0$$
$$(1-Q)\,Q = \ldots = 0$$

projectors have simple eigenvalues: just 1 and 0

$$Q|\vec{v}_i\rangle = q_i|\vec{v}_i\rangle \Rightarrow q_i|\vec{v}_i\rangle = Q|\vec{v}_i\rangle = Q^2|\vec{v}_i\rangle = q_i^2|\vec{v}_i\rangle$$

$$\Rightarrow q_i = q_i^2 \Rightarrow q_i = 1 \text{ or } 0$$

1.2. Representations in quantum mechanics

a) various representations

A physical system is characterized by a state $|\psi\rangle$ in a Hilbert space. "Observables" correspond to hermitean operators. Like using different coordinate systems and their sets of unit vectors to get representations of a given vector in a vector space, one can use different sets of basis states in the Hilbert space to get representations for $|\psi\rangle$. Frequently used sets of basis states are

① the coordinate-space states, i.e. eigenstates to the coordinate operator, e.g. in one dimension: the continuous set of states $|x\rangle$, $x \in \mathbb{R}$, with

$$\underset{\text{operator}}{\hat{x}}|x\rangle = \underset{\text{number}}{x}|x\rangle$$

normalization: $\langle x'|x\rangle = \delta(x'-x)$ (cf. $\langle \vec{e}_i|\vec{e}_j\rangle = \delta_{ij}$)

completeness: $\int dx\, |x\rangle\langle x| = \mathbb{1}$ (cf. $\sum_i |\vec{e}_i\rangle\langle\vec{e}_i| = \mathbb{1}$)

For a one-particle system in one dimension the quantity

$$\psi(x) := \langle x|\psi\rangle$$

is the well-known probability amplitude to find the particle at position x

② the momentum-space states (again in one dimension) $|p\rangle$:

$$\hat{p}\,|p\rangle = p\,|p\rangle$$

normalization (convention): $\langle p'|p\rangle = 2\pi\hbar\,\delta(p'-p)$

completeness: $\int \frac{dp}{2\pi\hbar}|p\rangle\langle p| = \mathbb{1}$

③ orbital angular-momentum states (three dimensions) $|l, m\rangle$:

$$(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)|l, m\rangle = \hat{L}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$\hat{L}_z |l, m\rangle = m\hbar |l, m\rangle$$

with $l = 0, 1, \ldots$; $m = -l, -l+1, \ldots, l-1, l$

④ for a particle with spin $\frac{1}{2}$ (irrespective of its position): just a two-dimensional vector space $|s\rangle$, $s = \uparrow, \downarrow$

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

⋮

To describe more complicated systems one can use product spaces and correspondingly direct products of states. For example, to describe two different particles, both with spin $\frac{1}{2}$ (e.g. the hydrogen atom), in three dimensions one can use $|\vec{r}_1, s_1, \vec{r}_2, s_2\rangle = |x_1, y_1, z_1, s_1, x_2, y_2, z_2, s_2\rangle$

$$= |x_1\rangle |y_1\rangle |z_1\rangle |s_1\rangle |x_2\rangle |y_2\rangle |z_2\rangle |s_2\rangle$$

normalization: $\langle \vec{r}_1', s_1', \vec{r}_2', s_2' | \vec{r}_1, s_1, \vec{r}_2, s_2\rangle$

$$= \langle x_1' | x_1\rangle \langle y_1' | y_1\rangle \langle z_1' | z_1\rangle \langle s_1' | s_1\rangle \langle x_2' | x_2\rangle \ldots$$

It can be useful to couple the spaces, e.g. to build center-of-mass and relative coordinates or to couple two spins to a total spin or to couple spin and orbital angular momentum to total angular momentum

b) the translation operator

We concentrate on the coordinate-space representation (for simplicity in one dimension).
In the following we only use the basic relation

$$[\hat{x}, \hat{p}] = i\hbar \quad (*)$$

and show that a unitary operation $e^{-iy\hat{p}/\hbar}$ causes a translation in space:

$$e^{-iy\hat{p}/\hbar}|x\rangle = |x+y\rangle$$

The important thing to stress is that $(*)$ is used without any particular representation, e.g. we do not use that $\hat{p}$ might have something to do with a derivative operator (in coordinate representation)!

We start with an infinitesimal small Δy. The state

$$|\varphi_x\rangle := e^{-i\Delta y\hat{p}/\hbar}|x\rangle \approx \left(\mathbb{1} - i\frac{\Delta y}{\hbar}\hat{p}\right)|x\rangle$$

is actually an eigen state of the coordinate operator:

$$\hat{x}|\varphi_x\rangle \approx \hat{x}\left(\mathbb{1} - i\frac{\Delta y}{\hbar}\hat{p}\right)|x\rangle = \hat{x}|x\rangle - i\frac{\Delta y}{\hbar}\hat{x}\hat{p}|x\rangle$$

$$= x|x\rangle - i\frac{\Delta y}{\hbar}\underbrace{[\hat{x},\hat{p}]}_{i\hbar}|x\rangle - i\frac{\Delta y}{\hbar}\hat{p}\hat{x}|x\rangle$$

$$= x|x\rangle + \Delta y|x\rangle - i\frac{\Delta y}{\hbar}x\hat{p}|x\rangle$$

$$= (x+\Delta y)\left(|x\rangle - \frac{i}{\hbar}\underbrace{\Delta y\frac{x}{x+\Delta y}}_{\approx\Delta y}\hat{p}|x\rangle\right)$$

$$\approx (x+\Delta y)\left(\mathbb{1} - \frac{i}{\hbar}\Delta y\hat{p}\right)|x\rangle \approx (x+\Delta y)|\varphi_x\rangle$$

$$\Rightarrow |\varphi_x\rangle = \underset{\substack{\uparrow \\ \text{normalisation}}}{N_x} |x+\Delta y\rangle$$

As a unitary operator $e^{-i\Delta y\hat{p}/\hbar}$ does not change the normalisation:

$$\langle\varphi_{x'}|\varphi_x\rangle = N_{x'}^* N_x \langle x'+\Delta y|x+\Delta y\rangle = N_{x'}^* N_x \delta(x'-x)$$

on the other hand

$$\langle\varphi_{x'}|\varphi_x\rangle = \left(e^{-i\ldots}|x'\rangle\right)^+ \left(e^{-i\ldots}|x\rangle\right) = \langle x'|e^{+i\ldots}e^{-i\ldots}|x\rangle$$

$\hat{p} = \hat{p}^+$

$$= \langle x'|x\rangle = \delta(x'-x)$$

$$\Rightarrow N_x = 1$$

$$\Rightarrow e^{-i\Delta y\hat{p}/\hbar}|x\rangle = |x+\Delta y\rangle \quad \text{for small } \Delta y$$

more general:

$$e^{-iy\hat{p}/\hbar}|x\rangle = \lim_{n\to\infty}\left(1-\frac{i}{\hbar}\hat{p}\frac{y}{n}\right)^n|x\rangle$$

each factor $\left(1-\frac{i}{\hbar}\hat{p}\frac{y}{n}\right)$ shifts position by $\frac{y}{n}$ for n large

$$\Rightarrow e^{-iy\hat{p}/\hbar}|x\rangle = \lim_{n\to\infty}|x+n\frac{y}{n}\rangle = |x+y\rangle$$

Correspondingly: $e^{iy\hat{p}/\hbar}\,\hat{x}\,e^{-iy\hat{p}/\hbar} = \hat{x} + y\mathbb{1}$

proof: apply both sides to arbitrary state $|x\rangle$

c) momentum eigen states in coordinate space

We shall show that the momentum eigen states are plane waves in coordinate space. Again we only use

$$[\hat{x},\hat{p}] = i\hbar$$

Define $f_p(x) = \langle x|p\rangle$ = representation of the momentum eigen state $|p\rangle$ in coordinate space

$$\hat{p}|p\rangle = p|p\rangle$$

$$\Rightarrow \langle x|\hat{p}|p\rangle = p\, f_p(x)$$

$$\Rightarrow \langle x|\hat{p}\int dx'|x'\rangle\langle x'|p\rangle = p\, f_p(x)$$

$$\Rightarrow \int dx' \langle x|\hat{p}|x'\rangle f_p(x') = p\, f_p(x)$$

Thus we need the momentum operator $\hat{p}$ in coordinate space representation, i.e. $\langle x|\hat{p}|x'\rangle$

start with $|x+\Delta x\rangle = e^{-i\Delta x\hat{p}/\hbar}|x\rangle \underset{\Delta x \text{ small}}{\approx} (\mathbb{1} - i\frac{\Delta x}{\hbar}\hat{p})|x\rangle$

$$\Rightarrow \delta(x'-x-\Delta x) = \langle x'|x+\Delta x\rangle \approx \langle x'|(\mathbb{1} - \frac{i}{\hbar}\Delta x\,\hat{p}|x\rangle$$

$$= \delta(x'-x) - \Delta x\,\frac{i}{\hbar}\langle x'|\hat{p}|x\rangle$$

$$\Rightarrow \langle x'|\hat{p}|x\rangle = \frac{\hbar}{i}\frac{1}{\Delta x}\left(\delta(x'-x) - \delta(x'-x-\Delta x)\right) \text{ for small } \Delta$$

$$= i\hbar\frac{1}{\Delta x}\left(\delta(x+\Delta x-x') - \delta(x-x')\right)$$

$$\approx i\hbar\frac{\partial}{\partial x}\delta(x-x')$$

$$= i\hbar\,\delta'(x-x')$$

or $\langle x|\hat{p}|x'\rangle = i\hbar\,\delta'(x'-x)$

$$\Rightarrow \int dx'\, i\hbar\, \delta'(x'-x)\, f_p(x') = p\, f_p(x)$$

$$\parallel$$

$$-i\hbar \int dx'\, \delta(x'-x)\, f_p'(x') = -i\hbar\, f_p'(x)$$

$\Rightarrow$ differential equation

$$f_p'(x) = \frac{ip}{\hbar} f_p(x)$$

$\Rightarrow f_p(x) = N_p e^{ipx/\hbar}$ plane wave

fix normalisation:

$$2\pi\hbar\delta(p-p') = \langle p|p'\rangle = \int dx \langle p|x\rangle\langle x|p'\rangle$$

$$= \int dx\, f_p^*(x)\, f_{p'}(x) = \int dx\, N_p^* N_{p'}\, e^{i(p'-p)x/\hbar}$$

$$= 2\pi\hbar\, \delta(p'-p)\, N_p^* N_{p'}$$

$\Rightarrow N_p = 1 \qquad \Rightarrow \langle x|p\rangle = e^{ipx/\hbar}$

d) changing the representation

e.g from coordinate space to momentum space

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle = \int dx\, e^{-ipx/\hbar}\, \psi(x)$$

and vice versa

$$\langle x|\psi\rangle = \int \frac{dp}{2\pi\hbar} \langle x|p\rangle\langle p|\psi\rangle = \int \frac{dp}{2\pi\hbar}\, e^{ipx/\hbar} \langle p|\psi\rangle$$