3.2. Dirac equation

In the following we want to construct a formalism for relativistic (non-interacting) spin-$\frac{1}{2}$ fermions
For convenience we set $c = 1 = \hbar$
One way to observe spin is by coupling the particles to an external electromagnetic field (e.g. Stern-Gerlach experiment). Therefore we will allow (if necessary) for such an interaction, but we do not consider here interactions among the particles

Non-relativistic many-body system (with only external electromagnetic interaction) for spin-$\frac{1}{2}$ fermions:

In chapter 2 we have most of the time neglected the spin degree of freedom. Now we consider it explicitly:

$\psi^+_{n.r.}(\vec{r}, \sigma; t)$ creates a state at position $\vec{r}$ with spin $+\frac{1}{2}/-\frac{1}{2}$ for $\sigma = +1/-1$ (at time t):

$$\psi^+_{n.r.}(\vec{r}, \pm 1; t)|0\rangle = |\vec{r}, s = \pm\tfrac{1}{2}; t\rangle$$

anticommutators:

$$\{\psi_{n.r.}(\vec{r}, \sigma; t), \psi_{n.r.}(\vec{r}\,', \sigma'; t)\} = 0 = \{\psi^+_{n.r.}(\vec{r}, \sigma; t), \psi^+_{n.r.}(\vec{r}\,', \sigma'; t)\}$$

$$\{\psi_{n.r.}(\vec{r}, \sigma; t), \psi^+_{n.r.}(\vec{r}\,', \sigma'; t)\} = \delta(\vec{r} - \vec{r}\,')\,\delta_{\sigma\sigma'}$$

with $\{A, B\} = AB + BA$

The electromagnetic field is expressed in terms of potentials: the electric potential A_0 (often called Φ) and the vector potential $\vec{A}$ with the field strengths given by

$\vec{E} = -\vec{\nabla} A_0 - \frac{\partial}{\partial t}\vec{A}$ electric field strength

$\vec{B} = \vec{\nabla} \times \vec{A} = \mathrm{rot}\,\vec{A}$ magnetic " "

The coupling of a point particle with charge e to the vector potential appears via "minimal substitution"

$$\vec{p} \rightarrow \vec{p} - e\vec{A}$$

For non-relativistic particles the electric potential A_0 contributes to the potential energy with eA_0

$\Rightarrow$ classical energy $E = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eA_0$

If the particle has a spin (which is a quantum effect and cannot be covered by the classical energy), there is an additional coupling of the magnetic field $\vec{B}$ to the spin (see below).

It is convenient to introduce the two-component object

$$\varphi_{n.r.}(\vec{r}, t) = \begin{pmatrix} \varphi_{n.r.}(\vec{r}, +1; t) \\ \varphi_{n.r.}(\vec{r}, -1; t) \end{pmatrix}$$

The Hamiltonian of the many-body system is given by (proof like in chapter 2)

$$H(t) = \int d^3r\, \varphi^+_{n.r.}(\vec{r}, t)\left[\frac{1}{2m}\left(\frac{1}{i}\vec{\nabla}_r - e\vec{A}(t, \vec{r})\right)^2 + g\mu\frac{1}{2}\vec{\sigma}\cdot\vec{B}(t, \vec{r}) + eA_0(t, \vec{r})\right]\varphi_{n.r.}(\vec{r}, t)$$

$A_0, \vec{A}$ can depend on t

with the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the magneton $\mu = \frac{\pm e}{2m}$ and the gyromagnetic factor g (for electrons: $g \approx 2$)

This leads to the equation of motion for the annihilation operator (the two-component object!)

$$i\frac{\partial}{\partial t}\varphi_{n.r.}(\vec{r},t) = \left[\frac{1}{2m}\left(\frac{1}{i}\vec{\nabla}_r - e\vec{A}\right)^2 + g\mu\frac{1}{2}\vec{\sigma}\cdot\vec{B} + eA_0\right]\varphi_{n.r.}(\vec{r},t) \quad (*)$$

(obtained from $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$)

This looks like the Pauli equation from one-body quantum mechanics – except for the fact that $\varphi_{n.r.}$ is an operator in Fock space and not a wave function.

We have written down (*) for two purposes:

1. After the introduction of the free Dirac equation, we will again couple the fermions to an external electromagnetic field. We will see that a non-relativistic approximation leads to (*) with the <u>prediction</u> $g = 2$ for point particles.
2. We use (*) as a motivation that the Dirac equation can be an equation for an object with several components.

Guess for relativistic equation (here without external fields): Dirac started with

$$i\frac{\partial}{\partial t}\psi(t,\vec{r}) = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla}_r + \beta m\right)\psi(t,\vec{r})$$

with an n-component object $\psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_n\end{pmatrix}$, the "Dirac spinor" and correspondingly constant $n\times n$ matrices $\alpha_x, \alpha_y, \alpha_z$ and β

Such an equation has at least the potential to treat space and time on equal footing since it is a first-order differential equation both in time and space

To be as close as possible to the non-relativistic case

$$i\frac{\partial}{\partial t}\varphi_{n.r.} = -\frac{1}{2m}\Delta\varphi_{n.r.}$$

Dirac demanded that the object, which acts on ψ on the right-hand side of his equation, is hermitian

$\Rightarrow \alpha_x^+ = \alpha_x, \ldots, \beta^+ = \beta$ (since $\hat{\vec{p}} = \frac{1}{i}\vec{\nabla}$ is hermitian)

In addition, each component of ψ should satisfy the Klein-Gordon equation to make sure that one gets a relativistic energy-momentum relation for the wave modes.

$$\Rightarrow -\frac{\partial^2}{\partial t^2}\psi = i\frac{\partial}{\partial t}\, i\frac{\partial}{\partial t}\psi = i\frac{\partial}{\partial t}\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

$$= \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

$$= \left(-\sum_{j,k}\alpha_j\alpha_k\nabla_j\nabla_k + \beta^2 m^2 + \frac{1}{i}\vec{\alpha}\cdot\vec{\nabla}\beta + \frac{1}{i}\beta\vec{\alpha}\cdot\vec{\nabla}\right)\psi$$

$$\stackrel{!}{=} \left(-\vec{\nabla}^2 + m^2\right)\psi$$

can be satisfied, if

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbb{1} \text{ for all } j,k \in \{x,y,z\},$$

$$\beta^2 = \mathbb{1},$$

$$\alpha_j\beta + \beta\alpha_j = 0.$$

Note: $\sum_{j,k}\alpha_j\alpha_k\nabla_j\nabla_k \stackrel{!}{=} \sum_j \nabla_j^2$ fixes only sum $\alpha_j\alpha_k + \alpha_k\alpha_j$

since $\sum\alpha_j\alpha_k\nabla_j\nabla_k = \sum\Big[\frac{1}{2}(\alpha_j\alpha_k + \alpha_k\alpha_j) + \underbrace{\frac{1}{2}(\alpha_j\alpha_k - \alpha_j\alpha_k)}_{\text{antisymmetric in exchange } j\leftrightarrow k}\Big]\underbrace{\nabla_j\nabla_k}_{\text{sym. in } j\leftrightarrow k}$

$$= \frac{1}{2}\sum(\alpha_j\alpha_k + \alpha_k\alpha_j)\nabla_j\nabla_k$$

From $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \mathbb{1}$ and hermiticity one can conclude that the eigen values are $+1$ or -1.

From

$$tr(\beta) = tr(\mathbb{1}\beta) = tr(\alpha_x \alpha_x \beta) \underset{\substack{\uparrow \\ \text{cyclicity} \\ \text{of trace}}}{=} \frac{1}{2} tr(\alpha_x \alpha_x \beta) + \frac{1}{2} tr(\alpha_x \beta \alpha_x)$$

$$= \frac{1}{2} tr\big(\alpha_x \underbrace{(\alpha_x \beta + \beta \alpha_x)}_{=0}\big) = 0$$

one concludes that n (the number of components of ψ) must be even, since $tr\,\beta$ is the sum of eigen values of β (which are ± 1).

For $n = 2$ one does not find enough (four) anticommuting matrices

$\Rightarrow$ $n = 4$ is smallest possible number

The choice for β and α_j is not unique. Following Dirac we choose

$$\beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

2×2 Pauli matrices

Exercise: Show that for this choice the equations at page $\boxed{12}$ are satisfied.

We now have the following tasks:

1. Find (number valued) solutions of

$$i \frac{\partial}{\partial t} f_k(t, \vec{r}) = \left(\frac{1}{i} \vec{\alpha} \cdot \vec{\nabla}_r + \beta m\right) f_k(t, \vec{r})$$

← index which enumerates solutions

2. Construct general solution for Fock-space operator

$$\psi(t, \vec{r}) = \sum_k a_k f_k(t, \vec{r})$$

4. Interpret action of a_k on Fock states, i.e. n-body states, and derive anticommutation relations for ψ components.
For this task a non-relativistic reduction might help.

5. Couple electromagnetic field to particles via minimal substitution, perform again non-relativistic reduction and derive Pauli-type equation.

3. Show that Dirac equation is indeed compatible with Lorentz invariance

task 1:

since we expect plane waves to solve the Dirac equation, we start with the ansatz

$$f(t,\vec{r}) = e^{-i(Et-\vec{p}\cdot\vec{r})}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

with two-component objects $\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ and $\chi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$

$$\Rightarrow E\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = (\vec{\alpha}\cdot\vec{p} + \beta m)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

right-hand side:

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \left[\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & 0\end{pmatrix} + \begin{pmatrix}m\mathbb{1} & 0\\ 0 & -m\mathbb{1}\end{pmatrix}\right]\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

$$= \begin{pmatrix}m\varphi + \vec{\sigma}\cdot\vec{p}\,\chi\\ \vec{\sigma}\cdot\vec{p}\,\varphi - m\chi\end{pmatrix}$$

$$\Rightarrow (E-m)\varphi = \vec{\sigma}\cdot\vec{p}\,\chi \quad \text{and} \quad (E+m)\chi = \vec{\sigma}\cdot\vec{p}\,\varphi$$

$$\Rightarrow \chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi \quad \text{and} \quad (E-m)\varphi = \vec{\sigma}\cdot\vec{p}\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi$$

$\rightsquigarrow$ have to solve equation for φ

$$\vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p} = \sum_{j,k}\sigma_j p_j \sigma_k p_k = \frac{1}{2}\sum \underbrace{\{\sigma_j,\sigma_k\}}_{=2\delta_{jk}} p_j p_k = \vec{p}^{\,2}$$

$\rightsquigarrow$ equation for φ:

$$(E-m)\,\varphi = \frac{\vec{p}^{\,2}}{E+m}\,\varphi$$

$$\Rightarrow (E^2-m^2)\,\varphi = \vec{p}^{\,2}\,\varphi$$

$\Rightarrow$ $E = \pm\sqrt{m^2+\vec{p}^{\,2}}$, φ arbitrary two-component object,

χ fixed by $\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\varphi$

$\rightsquigarrow$ for fixed $\vec{p}$ there are four independent solutions:

convention

$$f_{\vec{p},1}(t,\vec{r}) = e^{-i(E_p t-\vec{p}\cdot\vec{r})}\underbrace{\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}\sqrt{E_p+m}}_{=:\,u(\vec{p},+1)} = e^{-i(E_p t-\vec{p}\cdot\vec{r})}\,u(\vec{p},+1)$$

$$f_{\vec{p},2}(t,\vec{r}) = e^{-i(E_p t-\vec{p}\cdot\vec{r})}\underbrace{\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}\sqrt{E_p+m}}_{=:\,u(\vec{p},-1)} = e^{-i(E_p t-\vec{p}\cdot\vec{r})}\,u(\vec{p},-1)$$

$$f_{\vec{p},3}(t,\vec{r}) = e^{i(E_p t+\vec{p}\cdot\vec{r})}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{m-E_p}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}\sqrt{E_p+m}$$

$$f_{\vec{p},4}(t,\vec{r}) = e^{i(E_p t+\vec{p}\cdot\vec{r})}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{m-E_p}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}\sqrt{E_p+m}$$

with $E_p = +\sqrt{m^2+\vec{p}^{\,2}}$

One aspect is unsatisfying with the solutions $f_{\vec{p},3}$ and $f_{\vec{p},4}$:
The limit $\vec{p} = 0$ is not well-defined since both

$$\vec{\sigma}\cdot\vec{p} \to 0 \quad \text{and} \quad m - E_p \to 0$$

$\rightsquigarrow$ more convenient to start with arbitrary χ and fix φ from $(E-m)\varphi = \vec{\sigma}\cdot\vec{p}\,\chi$

$$\Rightarrow \varphi = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\chi \underset{\uparrow}{=} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi$$

$E = -E_p$ for solutions f_3, f_4

$$\rightsquigarrow \tilde{f}_{\vec{p},3}(t,\vec{r}) = e^{i(E_p t + \vec{p}\cdot\vec{r})} \underbrace{\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}\sqrt{E_p+m}}_{=: v(-\vec{p},+1)} = e^{i(E_p t + \vec{p}\cdot\vec{r})}\, v(-\vec{p},+1)$$

$$\tilde{f}_{\vec{p},4}(t,\vec{r}) = e^{i(E_p t + \vec{p}\cdot\vec{r})} \underbrace{\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{pmatrix}\sqrt{E_p+m}}_{=: v(-\vec{p},-1)} = e^{i(E_p t + \vec{p}\cdot\vec{r})}\, v(-\vec{p},-1)$$

In the limit $\vec{p} = 0$ we find now

$$f_{(\vec{p}=0),1}(t,\vec{r}) = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\sqrt{2m}, \quad f_{(\vec{p}=0),2}(t,\vec{r}) = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\sqrt{2m}$$

$$\tilde{f}_{(\vec{p}=0),3}(t,\vec{r}) = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\sqrt{2m}, \quad \tilde{f}_{(\vec{p}=0),4}(t,\vec{r}) = e^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\sqrt{2m}$$

Finally we note that $e^{i(E_p t+\vec{p}\cdot\vec{r})}$ corresponds to a wave with momentum $-\vec{p}$ instead of $\vec{p}$

⤳ use instead

$$g_{\vec{p},1}(t,\vec{r}) := f_{(-\vec{p}),3}(t,\vec{r}) = e^{i(E_p t-\vec{p}\cdot\vec{r})}\, v(\vec{p},+1)$$

$$g_{\vec{p},2}(t,\vec{r}) := f_{(-\vec{p}),4}(t,\vec{r}) = e^{i(E_p t-\vec{p}\cdot\vec{r})}\, v(\vec{p},-1)$$

⤳ $f_{\vec{p},1}$, $f_{\vec{p},2}$, $g_{\vec{p},1}$ and $g_{\vec{p},2}$ constitute a complete basis for the solutions of the Dirac equation

task 2:

$$\psi(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^3\, 2E_p}\Big(a(\vec{p},+1)\, f_{\vec{p},1}(t,\vec{r}) + a(\vec{p},-1)\, f_{\vec{p},2}(t,\vec{r})$$
$$+ b^+(\vec{p},+1)\, g_{\vec{p},1}(t,\vec{r}) + b^+(\vec{p},-1)\, g_{\vec{p},2}(t,\vec{r})\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2E_p} \sum_{s=\pm1} \Big(a(\vec{p},s)\, u(\vec{p},s)\, e^{-i(E_p t-\vec{p}\cdot\vec{r})} + b^+(\vec{p},s)\, v(\vec{p},s)\, e^{+i(E_p t-\vec{p}\cdot\vec{r})}\Big)$$

Similarities and differences to the Klein-Gordon case:

- Since the components of ψ are not hermitian, we have different operators a and b (and $u \neq v$ anyway)
- We have used b^+ instead of b since we expect that operators accompanied by the phase $e^{+i(E_p t-\vec{p}\cdot\vec{r})}$! have something to do with particle creation

ψ contains four different Fock space operators for a given momentum. This corresponds to four different particles, all with the same mass m. Without interactions one cannot distinguish these four particles. Coupling them to an electromagnetic field (see below), one finds that the particles are distinct by their spin orientation and by the sign of their charge. Convention: States annihilated by ψ are called "particles" (with spin up or down and with charge e), states created by ψ (via b) are called "anti-particles" (with spin up or down and with charge $(-e)$)

!

Note: It was unavoidable to have (at least) four components for ψ. This corresponds to the four different particles (look at $f_{\vec{p},\dots}$, $g_{\vec{p},\dots}$ for $\vec{p} = 0$).

$\Rightarrow$ prediction for existence of anti-particles task 3:

We will only start this here and then refer to the literature for the actual proof.

To treat time and space similarly we multiply the Dirac equation

$$i\frac{\partial}{\partial t}\psi = \frac{1}{i}\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m\psi$$

by β and define — these are indices, not exponents

$$\gamma^0 := \beta,\quad \gamma^1 = \beta\alpha_x,\quad \gamma^2 = \beta\alpha_y,\quad \gamma^3 = \beta\alpha_z$$

and $x := (t, \vec{r}) =: (x^0, x^1, x^2, x^3)$

$$\rightsquigarrow i\underbrace{\left(\gamma^0\frac{\partial}{\partial x^0} + \gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3}\right)}_{=\gamma\cdot\partial =: \not\partial}\psi - m\psi = 0$$

$\leftarrow$ read "d slash"

$\rightsquigarrow$ Dirac equation in "Lorentz covariant" form

$$(i\not\partial_x - m)\,\psi(x) = 0$$

It is easy to show that the γ^μ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{4\times4}$$

with the "metric tensor" $(g^{\mu\nu}) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

explicit form (Dirac-Pauli representation)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix},\quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix},\quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}$$

note: $\gamma^i \neq (\gamma^i)^\dagger$, instead $(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0$

Lorentz covariance of the Dirac equation means that for a given Lorentz transformation (change of coordinate system)

$$x \to x'$$

one finds a transformation

$$\psi(x) \to \psi'(x')$$

such that $\psi'(x')$ satisfies

$$(i \not\partial_{x'} - m)\, \psi'(x') = 0$$

For the proof we refer to the literature (e.g. Bjorken/Drell, Peskin/Schroeder, Brown, ...).

Note: We have followed Dirac's heuristic way to construct the Dirac equation. Instead (and formally more convincing) one can start from the representations of the Lorentz group which correspond to spin-$\frac{1}{2}$ objects. Constructing a proper, i.e. Lorentz covariant, equation of motion also leads to the Dirac equation, see, e.g., Brown, Peskin/Schroeder.

task 4:

Guided by our experience from the Klein-Gordon theory and remembering that we want to describe fermions here, we demand

$$\{a(\vec{p},s), a(\vec{p}',s')\} = 0 = \{a^+(\vec{p},s), a^+(\vec{p}',s')\}$$

$$\{a(\vec{p},s), a^+(\vec{p}',s')\} = (2\pi)^3\, 2E_p\, \delta(\vec{p}-\vec{p}')\, \delta_{ss'}$$

$$\{b(\vec{p},s), b(\vec{p}',s')\} = 0 = \{b^+(\vec{p},s), b^+(\vec{p}',s')\}$$

$$\{b(\vec{p},s), b^+(\vec{p}',s')\} = (2\pi)^3\, 2E_p\, \delta(\vec{p}-\vec{p}')\, \delta_{ss'}$$

$$\{a(\vec{p},s), b^{(+)}(\vec{p}',s')\} = 0 = \{a^+(\vec{p},s), b^{(+)}(\vec{p}',s')\}$$

$$a(\vec{p},s)|0\rangle = 0 = b(\vec{p},s)|0\rangle$$

$$a^+(\vec{p},s)|0\rangle = |\vec{p}, s, \text{particle}\rangle$$

$$b^+(\vec{p},s)|0\rangle = |\vec{p}, s, \text{antiparticle}\rangle$$

anticommutation relations for ψ and ψ^+:

from relations above it is obvious:

$$\{\psi(t,\vec{r}), \psi(t,\vec{r}')\} = 0 = \{\psi^+(t,\vec{r}), \psi^+(t,\vec{r}')\}$$

the non-trivial anticommutator is

$$\{\psi_j(t,\vec{r}), \psi_k^+(t,\vec{r}')\}$$
$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3p'}{(2\pi)^3 2E_{p'}} \sum_{s,s'} \Big(\{a(\vec{p},s), a^+(\vec{p}',s')\}\, u_j(\vec{p},s)\, u_k^+(\vec{p}',s')\, e^{-i(E_pt - E_{p'}t - \vec{p}\vec{r} + \vec{p}'\vec{r}')}$$
$$+ \{b^+(\vec{p},s), b(\vec{p}',s')\}\, v_j(\vec{p},s)\, v_k^+(\vec{p}',s')\, e^{+i(E_pt - E_{p'}t - \vec{p}\vec{r} + \vec{p}'\vec{r}')} \Big)$$

$$\Rightarrow \left\{ \psi_j(t,\vec{r}), \psi_k^+(t,\vec{r}\,') \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s \left(u_j(\vec{p},s)\, u_k^+(\vec{p},s)\, e^{i\vec{p}(\vec{r}-\vec{r}\,')} + v_j(\vec{p},s)\, v_k^+(\vec{p},s)\, e^{-i\vec{p}(\vec{r}-\vec{r}\,')} \right)$$

can change integration $\vec{p} \to -\vec{p}$ in second term

$\Rightarrow$ have to calculate $\sum_s \left(u_j(\vec{p},s)\, u_k^+(\vec{p},s) + v_j(-\vec{p},s)\, v_k^+(-\vec{p},s) \right)$

example: $j = k = 1$

$$\Rightarrow u_1(\vec{p},+1) = \sqrt{E_p+m},\ u_1(\vec{p},-1) = 0,\ v_1(-\vec{p},+1) = -\frac{1}{\sqrt{E_p+m}} (\vec{\sigma}\cdot\vec{p})_{11},$$

$$v_1(-\vec{p},-1) = -\frac{1}{\sqrt{E_p+m}} (\vec{\sigma}\cdot\vec{p})_{12}$$

$$\Rightarrow \sum_s \left(u_1(\vec{p},s)\, u_1^+(\vec{p},s) + v_1(-\vec{p},s)\, v_1^+(-\vec{p},s) \right)$$

$$= E_p + m + \underbrace{\frac{1}{E_p+m} (\vec{\sigma}\cdot\vec{p})_{11} (\vec{\sigma}\cdot\vec{p})_{11} + \frac{1}{E_p+m} (\vec{\sigma}\cdot\vec{p})_{12} (\vec{\sigma}\cdot\vec{p})_{21}}_{= \frac{1}{E_p+m} (\underbrace{\vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p}}_{=\vec{p}^2 \mathbb{1}_{2\times 2}})_{11} = \frac{\vec{p}^2}{E_p+m} = \frac{E_p^2 - m^2}{E_p+m} = E_p - m}$$

$$= 2E_p$$

a complete calculation shows:

$$\sum_s \left(u_j(\vec{p},s)\, u_k^+(\vec{p},s) + v_j(-\vec{p},s)\, v_k^+(-\vec{p},s) \right) = 2E_p\, \delta_{jk}$$

$$\Rightarrow \left\{ \psi_j(t,\vec{r}), \psi_k^+(t,\vec{r}\,') \right\} = \int \frac{d^3p}{(2\pi)^3 2E_p}\, 2E_p\, \delta_{jk}\, e^{i\vec{p}(\vec{r}-\vec{r}\,')}$$

$$= \delta(\vec{r}-\vec{r}\,')\, \delta_{jk}$$

Note: In principle we could have started from the Dirac equation and demanded commutation instead of anticommutation relations for the operators.
This would not lead to a meaningful theory (without proof).
Also starting with the Klein-Gordon equation and demanding anticommutation instead of commutation relations would not lead to a meaningful theory. This is the essence of the spin-statistics theorem.

task 5: Coupling to external electromagnetic field and non-relativ. limit

starting from the free Dirac equation

$$(i\not{\partial} - m)\psi = 0$$

we use minimal substitution (now for space and time)

$$E \to E - eA_0 \quad \Rightarrow \quad i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - eA_0$$

$$\vec{p} \to \vec{p} - e\vec{A} \quad \Rightarrow \quad \frac{1}{i}\vec{\nabla} \to \frac{1}{i}\vec{\nabla} - e\vec{A}$$

to obtain

$$\left[\gamma^0\left(i\frac{\partial}{\partial t} - eA_0\right) - \vec{\gamma}\left(\frac{1}{i}\vec{\nabla} - e\vec{A}\right) - m\right]\psi = 0$$

non-relativistic limit:
introduce again two-component "spinors" φ, χ with

$$\psi(t,\vec{r}) = e^{-imt}\begin{pmatrix}\varphi(t,\vec{r})\\ \chi(t,\vec{r})\end{pmatrix}$$

$$\rightsquigarrow \left[\begin{pmatrix} i\partial_t - eA_0 & 0 \\ 0 & -i\partial_t + eA_0\end{pmatrix} - \begin{pmatrix} 0 & -i\vec{\sigma}\cdot\vec{\nabla} - e\vec{\sigma}\cdot\vec{A} \\ i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A} & 0\end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\right]\begin{pmatrix}\varphi\\ \chi\end{pmatrix} e^{-imt} = 0$$

$$\Rightarrow \begin{cases} (i\partial_t - eA_0 - m)\varphi + (i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\chi + m\varphi = 0 \\ (-i\partial_t + eA_0 - m)\chi - (i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\varphi + m\chi = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (i\partial_t - eA_0)\varphi = -(i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\chi \\ (2m + i\partial_t - eA_0)\chi = -(i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A})\varphi \end{cases}$$

The idea is that the coefficient e^{-imt} has accounted for the fast oscillation (if all three-momenta and all external fields are small compared to m)

$\Rightarrow$ approximate: $(2m + i\partial_t - eA_0)\chi \approx 2m\chi$

$\Rightarrow \chi \approx -\frac{1}{2m}\left(i\vec{\sigma}\cdot\vec{\nabla} + e\vec{\sigma}\cdot\vec{A}\right)\varphi$ (note: χ is small, $O(\frac{1}{m})$)

$\Rightarrow$ equation for φ:

$$(i\partial_t - eA_0)\varphi \approx \frac{1}{2m}\left(i\vec{\sigma}\vec{\nabla} + e\vec{\sigma}\cdot\vec{A}\right)^2\varphi$$

$\rightsquigarrow$ calculate $(\vec{\sigma}\cdot\vec{\pi})^2$ for arbitrary object $\vec{\pi}$:

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \sum_{j,k}\sigma_j\sigma_k\pi_j\pi_k = \sum_{j,k}\Big(\frac{1}{2}\underbrace{[\sigma_j,\sigma_k]}_{=2i\sum_\ell \varepsilon_{jk\ell}\sigma_\ell} + \frac{1}{2}\underbrace{\{\sigma_j,\sigma_k\}}_{=2\delta_{jk}}\Big)\pi_j\pi_k$$

$$= i\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi}) + \vec{\pi}^2$$

for $\vec{\pi} = i\vec{\nabla} + e\vec{A}$:

$$(\vec{\pi}\times\vec{\pi})\varphi = (i\vec{\nabla}+e\vec{A})\times(i\vec{\nabla}+e\vec{A})\varphi = -\underbrace{\vec{\nabla}\times\vec{\nabla}}_{=0}\varphi + e^2\underbrace{\vec{A}\times\vec{A}}_{=0}\varphi + ie\vec{\nabla}\times\vec{A}\varphi + ie\vec{A}\times\vec{\nabla}\varphi$$

note that in $\vec{\nabla}\times\vec{A}\varphi$ the differential operator acts on $\vec{A}$ and φ

$$\Rightarrow \vec{\nabla}\times\vec{A}\varphi = (\vec{\nabla}\times\vec{A})\varphi - \vec{A}\times\vec{\nabla}\varphi$$

$$\Rightarrow (\vec{\pi}\times\vec{\pi})\varphi = ie(\vec{\nabla}\times\vec{A})\varphi = ie\vec{B}\varphi$$

$$\Rightarrow (\vec{\sigma}\cdot\vec{\pi})^2\varphi = -e\vec{\sigma}\cdot\vec{B}\varphi + \left(\frac{1}{i}\vec{\nabla} - e\vec{A}\right)^2\varphi$$

$\Rightarrow$ equation of motion for φ:

$$i\partial_t\varphi \approx \frac{1}{2m}\left(\frac{1}{i}\vec{\nabla} - e\vec{A}\right)^2\varphi - \frac{e}{2m}\vec{\sigma}\cdot\vec{B}\varphi + eA_0\varphi$$

This should be compared to the Pauli-type equation (page $\boxed{11}$)

$$\Rightarrow \underbrace{-\frac{e}{2m}}_{=\mu} = \frac{1}{2}g\mu \quad \Rightarrow \boxed{g = 2}$$

$\Rightarrow$ using the Dirac equation and the principle of minimal substitution leads to the correct gyromagnetic factor for a point particle