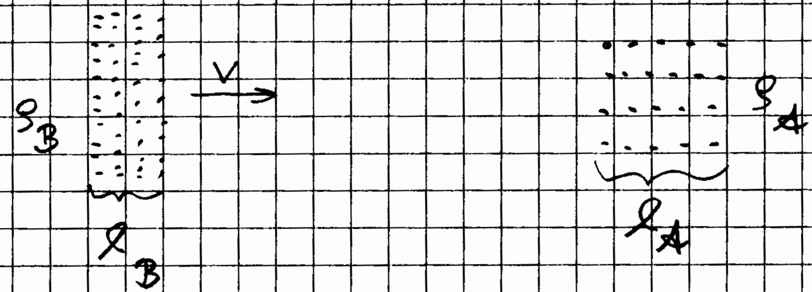
3.3. Scattering theory

a) Cross section

Consider target of particles of type A with density ρ_A at rest (number density, i.e. particles per volume - not: mass per volume) and a bunch of particles of type B with density ρ_B aiming at A with velocity v. Their overlap area is A.



The cross section should characterize a single scattering event. On the other hand it is intuitively clear that the number of scattering events is proportional to $\ell_A \ell_B \rho_A \rho_B A$

$\rightsquigarrow$ Definition:

$$\sigma = \frac{\text{number of scattering events}}{\ell_A \ell_B \rho_A \rho_B A}$$ scattering cross section

One can express the scattering cross section also in an alternative way:
introduce the time t which the beam needs to pass by:

$$t = \frac{\ell_B}{v};$$

the number of target particles is $\rho_A \ell_A A$;

the scattering rate R is the number of scattering events per target particle per time:

$$R = \frac{\text{number of scattering events}}{\rho_A \ell_A A t}$$

finally the <u>flux</u> of incident particles is:

$$\text{flux} = v \cdot \rho_B = \frac{\ell_B}{t}\rho_B = \frac{\text{number of streaming particles}}{(\text{time}) \cdot \left(\text{area perpendicular to streaming direction}\right)}$$

$$\Rightarrow \sigma = \frac{\text{rate}}{\text{flux}} = \frac{\text{number of scattering events}}{\rho_A \ell_A A \underbrace{t\, v}_{\ell_B} \rho_B}$$

b) S matrix

Obviously the cross section is related to the probability that collisions happen. We consider a reaction where two initial particles (of type A and B, respectively) scatter into n final particles. Typically the detectors measure the momenta of the final states. We denote them by $\vec{p}_f$, $f = 1, .., n$. The corresponding energy is $E_f = \sqrt{m_f^2 + \vec{p}_f^{\,2}}$ where m_f is the mass of the f'th final-state particle. We characterize the initial state by two wave packets for the particles of type A and B

$$|f_A\rangle := \int \frac{d^3k}{(2\pi)^3 2E_{A,k}} f_A(\vec{k})\, |\vec{k}\rangle$$

← wave packet

with $E_{A,k} = \sqrt{m_A^2 + \vec{k}^2}$

correspondingly for $|f_B\rangle$

The respective central momentum of the wave packet is $\vec{p}_A$ or $\vec{p}_B$.

Example: $f_A(\vec{k}) \sim \exp\left(-(\vec{k} - \vec{p}_A)^2 / \Delta k^2\right) \exp(-i \vec{x}_A \cdot \vec{k})$

up to normalisation

width of wave packet

location of wave packet

Normalization:

$$1 \stackrel{!}{=} \langle f_A | f_A \rangle = \int \frac{d^3k}{(2\pi)^3 2E_{A,k}} \frac{d^3k'}{(2\pi)^3 E_{A,k'}} \underbrace{\langle \vec{k}' | \vec{k} \rangle}_{=(2\pi)^3 2E_{A,k}\delta(\vec{k}-\vec{k}')} f_A^*(\vec{k}') f_A(\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_{A,k}} |f_A(\vec{k})|^2 \quad ; \text{ same for } B$$

Note: We use wave packets for the initial state and not plane waves to ensure a proper normalization.

The wave packets are prepared long before the reaction happens. The final momenta are measured much later than the reaction has happened. The probability that the initial particles scatter into n particles with arbitrary momenta p_f, $f = 1, \dots, n$, is given by (in the Heisenberg picture)

$$P(AB \to 12 \dots n) = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \left(\prod_{f=1}^{n} \int \frac{d^3p_f}{(2\pi)^3 2E_f} \right) |\langle p_1 \dots p_n ; t_f | f_A, f_B ; t_i \rangle|^2$$

If one fixes the final momenta to small volumes d^3p_f around $\vec{p}_f$, the differential probability is

$$\underline{dP} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \left(\prod_{f=1}^{n} \frac{d^3p_f}{(2\pi)^3 2E_f} \right) |\langle p_1, \dots, p_n ; t_f | f_A, f_B ; t_i \rangle|^2$$

$$= \int \left(\prod_{i=A,B} \frac{d^3k_i}{(2\pi)^3 2E_{i,k}} \frac{d^3\bar{k}_i}{(2\pi)^3 2E_{i,\bar{k}}} \right) f_A(\vec{k}_A) f_A^*(\vec{\bar{k}}_A) f_B(\vec{k}_B) f_B^*(\vec{\bar{k}}_B)$$

$$\cdot \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f} \right) \cdot S_{fi} \cdot S_{f\bar{i}}^*$$

with

$$S_{fi} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle p_1, \dots, p_n ; t_f | k_A, k_B ; t_i \rangle$$

and $S_{f\bar{i}}$ obtained from S_{fi} by $k \to \bar{k}$

We know already that the evolution in time is governed by a unitary operation

$$\Rightarrow \quad S_{fi} = \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle p_1, \ldots, p_n ; t_f | k_A, k_B ; t_i \rangle =: \underbrace{\langle p_1, \ldots, p_n | S | k_A, k_B \rangle}_{\text{now time independent}}$$

with a unitary operator S in Fock space, i.e. $S^+ S = \mathbb{1}$

S is called "S matrix"

If no interaction happens, S is just unity. (Then all matrix elements vanish for $n \neq 2$.) To isolate true interactions one introduces the "T matrix":

$$S = \mathbb{1} + i T$$

from $S^+ S = \mathbb{1}$ one obtains

$$-i T^+ + i T + T^+ T = 0$$

$$\Rightarrow -2 \,\mathrm{Im}\, T = T^+ T \qquad \text{unitarity relation}$$

$$\text{with } \mathrm{Im}\, T = \frac{T - T^+}{2i}$$

In principle S and T can be calculated, if the Hamiltonian and in particular its interaction part is known.
In practice one often uses perturbation theory to determine T approximately (see e.g. the lecture on quantum field theory).
In the following we assume that T is known and relate it to the scattering cross section.

In an elementary reaction (in the absence of external fields or additional initial particles) total energy and total momentum are conserved. It is useful to define the "invariant matrix element" or "Feynman matrix element" $\mathcal{M}$ via

$$\langle p_1,\dots,p_n | T | k_A, k_B\rangle = (2\pi)^4 \underbrace{\delta(k_A + k_B - \sum_f p_f)}_{=\delta(E_{A,k_A} + E_{B,k_B} - \sum_f E_f)\,\delta(\vec{k}_A + \vec{k}_B - \sum_f \vec{p}_f)} \mathcal{M}(k_A k_B \to \{p_f\})$$

Also for the cross section we want that reactions take place. Therefore it is sufficient to keep iT and drop $\mathbb{1}$ (in $S = \mathbb{1} + iT$) in the calculation of the reaction probability

$$dP = \int \left(\prod_{i=A,B} \frac{d^3k_i}{(2\pi)^3 2E_{i,k}} \frac{d^3\bar{k}_i}{(2\pi)^3 2E_{i,\bar{k}}} \right) f_A(\vec{k}_A) f_A^*(\vec{\bar{k}}_A) f_B(\vec{k}_B) f_B^*(\vec{\bar{k}}_B)$$
$$\cdot \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f} \right) (2\pi)^8 \delta(k_A + k_B - \sum_f p_f)\, \delta(\bar{k}_A + \bar{k}_B - \sum_f p_f)\, \mathcal{M}(k_A k_B \to \{p_f\}) \mathcal{M}^*(\bar{k}_A \bar{k}_B \to \{p_f\})$$

Now we have to specify our wave packets and make contact with the cross section. Ideally we are interested in initial particles with fixed momenta, i.e. in plane waves, but we have to use wave packets to obtain normalized/normalizable states
$\rightsquigarrow$ use narrow wave packets such that one can replace k_A, k_B by p_A, p_B wherever possible

$$\rightsquigarrow \delta(k_A + k_B - \sum_f p_f)\, \delta(\bar{k}_A + \bar{k}_B - \sum_f p_f)\, \mathcal{M}(k_A k_B \to \{p_f\}) \mathcal{M}^*(\bar{k}_A \bar{k}_B \to \{p_f\})$$
$$= \delta(k_A + k_B - \bar{k}_A - \bar{k}_B)\, \delta(\bar{k}_A + \bar{k}_B - \sum_f p_f)\, \mathcal{M}(k_A k_B \to \{p_f\}) \mathcal{M}^*(\bar{k}_A \bar{k}_B \to \{p_f\})$$
$$\approx \delta(k_A + k_B - \bar{k}_A - \bar{k}_B)\, \delta(p_A + p_B - \sum_f p_f)\, |\mathcal{M}(p_A p_B \to \{p_f\})|^2$$

Besides the small width of the wave packets the hope is that the cross section does <u>not</u> depend on details of the shape of the wave packets.

one gets

$$d\mathcal{P} = \int \left(\prod_{i=A,B} \frac{d^3 k_i}{(2\pi)^3 2E_i} \frac{d^3 \bar{k}_i}{(2\pi)^3 2E_i} \right) f_A(\vec{k}_A) f_A^*(\vec{\bar{k}}_A) f_B(\vec{k}_B) f_B^*(\vec{\bar{k}}_B)$$

$$\cdot (2\pi)^4 \delta(k_A + k_B - \bar{k}_A - \bar{k}_B) \, dw$$

where

$$dw = \left(\prod_{f=1}^{n} \frac{d^3 p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta(p_A + p_B - \sum_f p_f) \left| \mathcal{M}(p_A p_B \to \{p_f\}) \right|^2$$

does not depend on the details of the wave packets.

Introducing $\psi_A(x) = \int \frac{d^3 k}{(2\pi)^3 2E_A} f_A(\vec{k}) \, e^{ikx}$

$(kx = E_k x_0 - \vec{k}\vec{x})$ and same for B

and rewriting

$$(2\pi)^4 \delta(k_A + k_B - \bar{k}_A - \bar{k}_B) = \int d^4x \, e^{ix(k_A + k_B - \bar{k}_A - \bar{k}_B)}$$

one gets

$$d\mathcal{P} = \int d^4x \, |\psi_A(x)|^2 \, |\psi_B(x)|^2 \, dw$$

Next we have to get closer to the situation for which the cross section is defined:

The wave packets f_A and f_B are strongly peaked around $\vec{p}_A$ and $\vec{p}_B$, respectively. Thus the functions ψ_A and ψ_B have large extensions in space (and time). It is basically the extension of target and beam, respectively. The density, e.g. in the target, is given by

$$\rho_A = \frac{|\psi_A(x)|^2}{\int d^3x \, |\psi_A(x)|^2}$$

same for B.

Note that the density is normalized to 1 since we consider one target (and one projectile) particle.

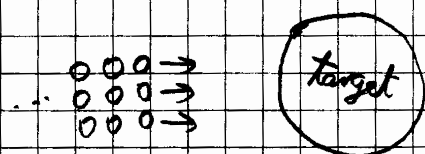
We assume now that the densities ρ_A and ρ_B are (more or less) constant within the wave packets except for the edges

$$\Rightarrow |\psi_A(x)| \approx \begin{cases} 0 & \text{outside of wave packet} \\ \text{const} =: \psi_A(\text{inside}) & \text{inside of wave packet} \end{cases}$$

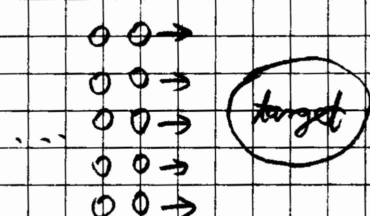
and same for $|\psi_B|^2$.

In addition we assume that during the scattering time the projectile wave function completely covers the target. Otherwise one would not fully probe the target properties. A classical example from hard-body scattering might illustrate this point:

situation 1:



situation 2:



Only in situation 2 the whole target is explored.

Finally we assume that $\vec{p}_A$ and $\vec{p}_B$ are anti-parallel (choose z direction)

With these assumptions one can calculate

$$dP = \int d^4x\, |\psi_A(x)|^2 |\psi_B(x)|^2\, dw$$

$$\approx \underset{\uparrow}{T} \int d^3x\, |\psi_A(x)|^2 |\psi_B(x)|^2\, dw \approx T\, |\psi_B(\text{inside})|^2 \int d^3x\, |\psi_A(x)|\, dw$$

scattering time, only then wave packets overlap

$\Rightarrow$ differential scattering rate $\frac{dP}{T} = |\psi_B(\text{inside})|^2 \int d^3x\, |\psi_A(x)|^2\, dw$

remember: $d\sigma = \frac{\text{scattering rate}}{\text{flux}}$

$$\text{flux} = |v_A - v_B|\, \rho_B = |v_A - v_B|\, \frac{|\psi_B(\text{inside})|}{\int d^3x\, |\psi_B(x)|^2}$$

relative velocity of projectile to target: $v_A = \frac{P_{Az}}{E_A}$, same for B $\left(E_A = \sqrt{m_A^2 + \vec{P}_A^{\,2}},\ \vec{P}_A = (0, 0, P_{Az})\right)$

$$\Rightarrow d\sigma = \frac{1}{|v_A - v_B|} \int d^3x\, |\psi_A(x)|^2 \int d^3x'\, |\psi_B(x')|^2\, dw$$

$\rightsquigarrow$ have to determine normalization integral

$$\int d^3x\, |\psi_A(x)|^2 = \int d^3x \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3k'}{(2\pi)^3 2E_{k'}} f_A(\vec{k})\, f_A^*(\vec{k}')\, e^{i(E_k - E_{k'})t} \cdot e^{-i(\vec{k}\vec{x} - \vec{k}'\vec{x})}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3k'}{(2\pi)^3 2E_{k'}} f_A(\vec{k})\, f_A^*(\vec{k}')\, e^{i(E_k - E_{k'})t} (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{1}{2E_k} |f_A(\vec{k})|^2$$

$$\approx \frac{1}{2E_A} \int \frac{d^3k}{(2\pi)^3 2E_k} |f_A(\vec{k})|^2 = \frac{1}{2E_A}$$

$E_k \approx E_A$ if f_A strongly peaked around $\vec{k} = \vec{P}_A$ $\left(E_A = \sqrt{m_A^2 + \vec{P}_A^{\,2}}\right)$

see page $\boxed{23}$, top

$\Rightarrow$ finally

$$d\sigma = \frac{1}{4E_A E_B |v_A - v_B|} \underbrace{\left(\prod_{f=1}^{n} \frac{d^3p_f}{(2\pi)^3 2E_f}\right) (2\pi)^4 \delta\Big(p_A + p_B - \sum_f p_f\Big) \left|\mathcal{M}(p_A p_B \to \{p_f\})\right|^2}_{\text{Lorentz invariant}}$$

Remarks:

- All dependence on the wave packets is gone
- The cross section is defined originally for the target at rest. It has the dimension of an area. The intuitive picture is an area perpendicular to the beam direction. Scattering happens, if the shortest distance between the scattering partners is smaller than $\sqrt{\frac{\sigma}{4\pi}}$.
- With respect to Lorentz transformations σ does not change if one boosts to frames which move along the beam axis. $\Rightarrow$ σ is the same for a) target at rest, b) beam at rest, c) center-of-mass frame ($\vec{p}_A + \vec{p}_B = 0$)

  Note that target and beam appear in completely symmetric form in cross section formula.

- In the frames where σ does not change, i.e. for $\vec{p}_A \parallel (-\vec{p}_B)$ or $\vec{p}_A \parallel \vec{p}_B$ one can write down a Lorentz invariant expression for σ:

  consider

$$s := (p_A + p_B)^2 = m_A^2 + m_B^2 + 2 p_A p_B$$

$$= m_A^2 + m_B^2 + 2E_A E_B - 2 E_A v_A E_B v_B$$

$$\Rightarrow \left(s - (m_A + m_B)^2\right)\left(s - (m_A - m_B)^2\right)$$

$$= \left[2E_A E_B (1 - v_A v_B) - 2 m_A m_B\right]\left[2E_A E_B(1 - v_A v_B) + 2 m_A m_B\right]$$

$$= 4 E_A^2 E_B^2 (1 - v_A v_B)^2 - 4 m_A^2 m_B^2$$

$$= 4 E_A^2 E_B^2 \underbrace{\left[(1 - v_A v_B)^2 - (1 - v_A^2)(1 - v_B^2)\right]}_{-2 v_A v_B + v_A^2 + v_B^2 = (v_A - v_B)^2 = |v_A - v_B|^2}$$

$m^2 = E^2 - p^2 = E^2(1 - v^2)$

$$\Rightarrow 2\sqrt{\left(s - (m_A + m_B)^2\right)\left(s - (m_A - m_B)^2\right)} = 4 E_A E_B |v_A - v_B|$$

c) simple examples:

consider two-body scattering

$p_1\, p_2 \to p_3\, p_4$

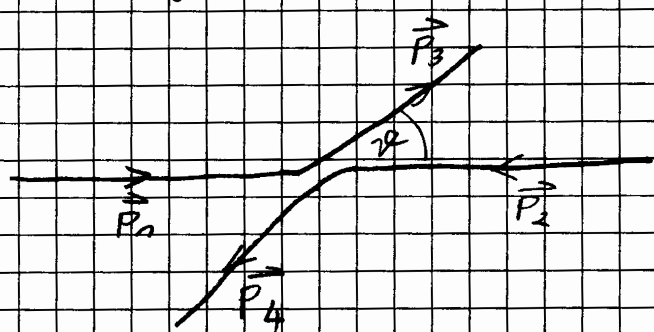
how many independent (and relevant) variables does one have?

⇝ two answers:

(1) recall classical case of scattering:

It is most economic to work in the center-of-mass frame (since center-of-mass shows free motion in the absence of external forces): $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = 0$

scattering is specified by momentum $|\vec{p}_1| = |\vec{p}_2|$ and scattering angle ϑ:



⇝ two relevant variables

(2) formal consideration:

the quantity which we want to calculate is Lorentz invariant

⇒ the only quantities which can appear are

$p_1 \cdot p_2,\ p_1 \cdot p_3,\ p_1 \cdot p_4,\ p_2 \cdot p_3,\ p_2 \cdot p_4,\ p_3 \cdot p_4$

However, p_4 is fixed by energy-momentum conservation

$p_4 = p_1 + p_2 - p_3$

⇝ three quantities $p_1 \cdot p_2,\ p_1 \cdot p_3,\ p_2 \cdot p_3$

In addition: $p_4^2 = m_4^2 \Rightarrow (p_1 + p_2 - p_3)^2 = m_4^2$

$\Bigg\}$ ⇒ two relevant variables

It is convenient to introduce the Mandelstam variables

$$s := (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t := (p_1-p_3)^2 = (p_2-p_4)^2$$
$$u := (p_1-p_4)^2 = (p_2-p_3)^2$$

They are related by

$$\begin{aligned} s+t+u &= (p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2 \\ &= m_1^2 + 2p_1\cdot p_2 + m_2^2 + m_1^2 - 2p_1\cdot p_3 + m_3^2 + m_1^2 - 2p_1\cdot p_4 + m_4^2 \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + \underbrace{2p_1\cdot\underbrace{(p_2-p_3-p_4)}_{=-p_1}}_{=-2m_1^2} \\ &= m_1^2+m_2^2+m_3^2+m_4^2 \end{aligned}$$

⇝ can use, e.g., s and t as independent variables

In the center-of-mass frame one has

$$s = (p_1+p_2)^2 \underset{\vec{p}_1+\vec{p}_2=0}{\uparrow}{=} (E_1+E_2)^2 \underset{|\vec{p}_1|=|\vec{p}_2|}{\uparrow}{=} \left(\sqrt{m_1^2+\vec{p}_1^{\,2}} + \sqrt{m_2^2+\vec{p}_1^{\,2}}\right)^2$$

⇒ $\sqrt{s}$ is total energy and $|\vec{p}_1|$ can be expressed in terms of s and vice versa

$$t = (p_1-p_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2|\vec{p}_1||\vec{p}_3|\cos\vartheta$$

since $s = (p_3+p_4)^2 \underset{\substack{\vec{p}_3=-\vec{p}_4 \\ \Rightarrow |\vec{p}_3|=|\vec{p}_4|}}{\uparrow}{=} \left(\sqrt{m_3^2+\vec{p}_3^{\,2}} + \sqrt{m_4^2+\vec{p}_3^{\,2}}\right)^2$

one can express $|\vec{p}_3|$ and E_3 in terms of s

⇒ can express ϑ in terms of s and t

Suppose $\mathcal{M}$ is just a constant

$$\Rightarrow d\sigma = \frac{1}{4E_1 E_2 |v_1 - v_2|} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) |\mathcal{M}|^2$$

⇝ for calculation of total cross section one has to determine the "two-body phase space"

$$\Pi_2 := \int \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$$

This quantity is Lorentz invariant, i.e. it has the same value in any frame

⇝ choose center-of-mass frame:

$$E_1 + E_2 = \sqrt{s}, \quad \vec{p}_1 + \vec{p}_2 = 0$$

$$\Rightarrow \Pi_2 = \int \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(\sqrt{s} - E_3 - E_4)\, \delta(\vec{p}_3 + \vec{p}_4)$$

$$= \frac{1}{(2\pi)^2} \int \frac{d^3p_3}{2E_3} \frac{1}{2E_{4,3}} \delta(\sqrt{s} - E_3 - E_{4,3})$$

$$E_{4,3} = \sqrt{m_4^2 + \vec{p}_4^{\,2}} = \sqrt{m_4^2 + \vec{p}_3^{\,2}}$$

The integrand only depends on $|\vec{p}_3|$

⇝ use spherical coordinates

$$\Pi_2 = \frac{1}{(2\pi)^2} 4\pi \int_0^\infty dp_3 \frac{p_3^2}{4E_3 E_{4,3}} \delta(\sqrt{s} - E_3 - E_{4,3})$$

instead of integrating over p_3 one might integrate over E_3

$$E_3^2 = m_3^2 + p_3^2 \Rightarrow E_3\, dE_3 = p_3\, dp_3$$

$$E_{4,3} = \sqrt{m_4^2 + p_3^2} = \sqrt{m_4^2 + E_3^2 - m_3^2}$$

$$\rightsquigarrow \Pi_2 = \frac{1}{4\pi} \int_{m_3}^\infty dE_3 \sqrt{E_3^2 - m_3^2} \frac{1}{E_{4,3}} \delta(\sqrt{s} - E_3 - E_{4,3})$$

The δ-function fixes E_3 to the following value:

$$\sqrt{s} - E_3 - E_{4,3} = 0$$

$$\Rightarrow \sqrt{s} - E_3 = E_{4,3}$$

$$\Rightarrow (\sqrt{s} - E_3)^2 = m_4^2 - m_3^2 + E_3^2$$

$$\Rightarrow s - 2\sqrt{s}\,E_3 = m_4^2 - m_3^2$$

$$\Rightarrow E_3 = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}$$

$$\Rightarrow E_{4,3} = \sqrt{s} - E_3 = \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}$$

the E_3 integration starts at m_3 ⇝ the δ-function is non-vanishing only, if $E_3 \geq m_3$

$$\Leftrightarrow s + m_3^2 - m_4^2 \geq 2\sqrt{s}\,m_3$$

$$\Leftrightarrow (\sqrt{s} - m_3)^2 \geq m_4^2$$

$$\Rightarrow \sqrt{s} \geq m_3 + m_4$$

This makes sense: The scattering can only happen, if one has enough energy. At least one needs the rest energies = masses of the final-state particles.

The momentum of particles 3 and 4 in the center-of-mass frame is

$$p_{cm,f} = \sqrt{E_3^2 - m_3^2} = \sqrt{(s + m_3^2 - m_4^2)^2 - 4 s m_3^2}\;\frac{1}{2\sqrt{s}}$$

$$= \underbrace{\sqrt{(s - (m_3+m_4)^2)(s - (m_3 - m_4)^2)}}_{\text{symmetric form in 3 and 4}}\;\frac{1}{2\sqrt{s}}$$

evaluation of δ-function: $\frac{\partial}{\partial E_3}(\sqrt{s} - E_3 - E_{4,3}) = -1 - \frac{2E_3}{2E_{4,3}} = -\frac{1}{E_{43}}(E_3 + E_{4,3})$

$$\Rightarrow \delta(\sqrt{s} - E_3 - E_{4,3}) = \frac{E_{43}}{E_3 + E_{4,3}}\,\delta(\sqrt{s} - E_3) = \frac{E_{43}}{\sqrt{s}}\,\delta(\sqrt{s} - E_3)$$

$$\Rightarrow \Pi_2 = \frac{1}{4\pi} \int_{m_3}^{\infty} dE_3 \sqrt{E_3^2 - m_3^2}\, \frac{1}{\sqrt{s}}\, \delta(\sqrt{s} - E_3)$$

$$= \frac{1}{4\pi} \frac{p_{cm,f}}{\sqrt{s}}\, \Theta(\sqrt{s} - m_3 - m_4)$$

$$\Rightarrow \sigma = \frac{1}{4E_1 E_2 |v_1 - v_2|}\, \Pi_2\, |\mathcal{M}|^2 \quad \text{if } \mathcal{M} \text{ constant}$$

we recall (page $\boxed{34}$):

$$4E_1 E_2 |v_1 - v_2| = 2\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}$$

comparing this expression with $p_{cm,f}$ we find

$$4E_1 E_2 |v_1 - v_2| = 4\sqrt{s}\, p_{cm,i}$$

with the momentum of particles 1 and 2 in the center-of-mass frame:

$$p_{cm,i} = \sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}\, \frac{1}{2\sqrt{s}}$$

$$\Rightarrow \sigma = \frac{1}{4\sqrt{s}\, p_{cm,i}}\, \frac{1}{4\pi}\, \frac{p_{cm,f}}{\sqrt{s}} |\mathcal{M}|^2 \quad \text{for } \sqrt{s} \geq m_3 + m_4$$

$$= \frac{1}{16\pi s}\, \frac{p_{cm,f}}{p_{cm,i}} |\mathcal{M}|^2$$

for elastic scattering: $p_{cm,f} = p_{cm,i}$

$$\Rightarrow \sigma = \frac{1}{16\pi s} |\mathcal{M}|^2$$

All this holds, if particles 3 and 4 are distinguishable.
If they are <u>indistinguishable</u> one counts all events twice by integrating $\vec{p}_3$ and $\vec{p}_4$ over all momenta.
In this case

$$\sigma = \frac{1}{32\pi s}\, \frac{p_{cm,f}}{p_{cm,i}} |\mathcal{M}|^2$$