Lagrangefunktion für kl. Mechanik (Skript 3.4.1)

Variations:

$$S[\vec{x}] = \int_{t_1}^{t_2} dt\; L(\vec{x}, \dot{\vec{x}}, t)$$

Hamiltonsches Prinzip: $\vec{x} = \vec{x}(t)$; $\vec{x}(t_1), \vec{x}(t_2)$ fest

$$\delta S = 0$$

Bew-Gl. mit konservativer Kraft

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x}) = -\begin{pmatrix} \partial_1 V \\ \partial_2 V \\ \partial_3 V \end{pmatrix}$$

$\vec{x} \to \vec{x} + \delta\vec{x}$; $\delta\vec{x}(t_1) = \delta\vec{x}(t_2) = 0$; $\delta t = 0$

$$\delta S = \int_{t_1}^{t_2} dt \left[\delta\vec{x}\cdot\frac{\partial L}{\partial \vec{x}} + \delta\dot{\vec{x}}\,\frac{\partial L}{\partial \dot{\vec{x}}}\right]$$

$$= \int_{t_1}^{t_2} dt \left[\delta\vec{x}\cdot\frac{\partial L}{\partial \vec{x}} - \delta\vec{x}\cdot\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}}\right] + \cancel{\delta\vec{x}\,\frac{\partial L}{\partial \dot{\vec{x}}}\Bigg|_{t=t_1}^{t=t_2}}$$

$$= \int_{t_1}^{t_2} dt\; \delta\vec{x}\cdot\underbrace{\left[\frac{\partial L}{\partial \vec{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}}\right]}_{\equiv 0} \stackrel{!}{=} 0 \text{ für all } \delta\vec{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{x}} \quad \text{Euler-Lagrange-Gln.}$$

$$m\ddot{\vec{x}} = -\frac{\partial V}{\partial \vec{x}} \Rightarrow \frac{\partial L}{\partial \vec{x}} = -\frac{\partial V}{\partial \vec{x}} \Rightarrow L(\vec{x}, \dot{\vec{x}}, t) = L_0(\dot{\vec{x}}, t) - V(\vec{x})$$

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L_0}{\partial \dot{\vec{x}}} \stackrel{!}{=} m\dot{\vec{x}} = \vec{p} \Rightarrow L_0 = \frac{m}{2}\dot{\vec{x}}^2$$

$$\boxed{L = \frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x})} \qquad L = T - V(\vec{x})$$ ②

$$T = \frac{m}{2}\dot{\vec{x}}^2 \text{ kin. Energie}$$

Beliebige generalisierte Koordinaten

$$\vec{x} = \vec{x}(q_1, \dots, q_k)$$

$$L = T - V = L(q, \dot{q}, t)$$

$$V = V[\vec{x}(q)] \equiv V(q)$$

$$T = \frac{m}{2}\dot{\vec{x}}^2$$

$$\dot{\vec{x}} = \dot{q}_k \frac{\partial \vec{x}}{\partial q_k} \quad \text{Summenkonvention}$$

$$\dot{\vec{x}}^2 = \dot{q}_k \frac{\partial \vec{x}}{\partial q_k} \cdot \dot{q}_l \frac{\partial \vec{x}}{\partial q_l} = g_{kl}(q)\, \dot{q}_k \dot{q}_l$$

$$g_{kl}(q) = \frac{\partial \vec{x}}{\partial q_k} \cdot \frac{\partial \vec{x}}{\partial q_l} = g_{lk}(q)$$

$$L = \frac{m}{2} g_{kl}(q) \dot{q}_k \dot{q}_l - V(q)$$

Variationsprinzip $\Rightarrow$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}$$

$$\frac{d}{dt}\Big[\underbrace{g_{kl}(q)\dot{q}_l m}_{p_k}\Big] = \frac{m}{2}\dot{q}_l \dot{q}_m \frac{\partial}{\partial q_k} g_{lm}(q) - \frac{\partial V}{\partial q_k}$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \text{(generalisierter Impuls)}$$

Äquivalente Lagrange-Funktionen (Skript 3.4.2)

$$L'(q,\dot q,t) = L(q,\dot q,t) + \underbrace{\Lambda(q,\dot q,t)}$$

$\Rightarrow$ sollen EL-Gln. ergeben

$$\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot q_k} = \frac{\partial \Lambda}{\partial q_k}$$ muß für alle $q(t)$ erfüllt

$$\dot q_\ell \frac{\partial^2\Lambda}{\partial q_\ell\,\partial\dot q_k} + \ddot q_\ell \underbrace{\frac{\partial^2\Lambda}{\partial\dot q_\ell\,\partial\dot q_k}}_{\overset{!}{=}0} + \frac{\partial^2\Lambda}{\partial\dot q_k\,\partial t} = \frac{\partial\Lambda}{\partial q_k}$$

$$\frac{\partial^2\Lambda}{\partial\dot q_\ell\,\partial\dot q_k}=0 \Rightarrow \frac{\partial\Lambda}{\partial\dot q_k} = \Lambda_{0k}(q,t)$$

$$\Rightarrow \Lambda = \dot q_k\,\Lambda_{0k}(q,t)+\Lambda_1(q,t)$$

$$\frac{\partial\Lambda}{\partial\dot q_k}=\Lambda_{0k}(q,t);\quad \frac{\partial}{\partial q_\ell}\frac{\partial\Lambda}{\partial\dot q_k} = \frac{\partial\Lambda_{0k}(q,t)}{\partial q_\ell}$$

$$\frac{\partial}{\partial t}\frac{\partial\Lambda}{\partial\dot q_k} = \frac{\partial}{\partial t}\Lambda_{0k}(q,t)$$

$$\dot q_\ell\,\frac{\partial\Lambda_{0k}(q,t)}{\partial q_\ell} + \frac{\partial\Lambda_{0k}(q,t)}{\partial t} = \dot q_\ell\,\frac{\partial\Lambda_{0\ell}(q,t)}{\partial q_k} + \frac{\partial\Lambda_1(q,t)}{\partial q_k}$$

$$\dot q_\ell\underbrace{\left(\frac{\partial\Lambda_{0k}}{\partial q_\ell}-\frac{\partial\Lambda_{0\ell}}{\partial q_k}\right)}_{\equiv 0} = -\frac{\partial\Lambda_{0k}}{\partial t}+\frac{\partial\Lambda_1}{\partial q_k}\overset{!}{\equiv}0$$

$$\frac{\partial}{\partial t}\Lambda_{0k} = \frac{\partial\Lambda_1}{\partial q_k} \Rightarrow \Lambda_{0k} = \frac{\partial\Omega}{\partial q_k}$$

$$\frac{\partial\Lambda_1}{\partial q_k} = \frac{\partial}{\partial t}\frac{\partial\Omega}{\partial q_k} \Rightarrow \Lambda_1 = \frac{\partial\Omega}{\partial t}$$

$$\Lambda = \dot{q}_{\alpha} \frac{\partial \Omega(q,t)}{\partial q_{\alpha}} + \frac{\partial \Omega(q,t)}{\partial t} = \frac{d}{dt}\Omega(q,t)$$

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}\Omega(q,t)$$