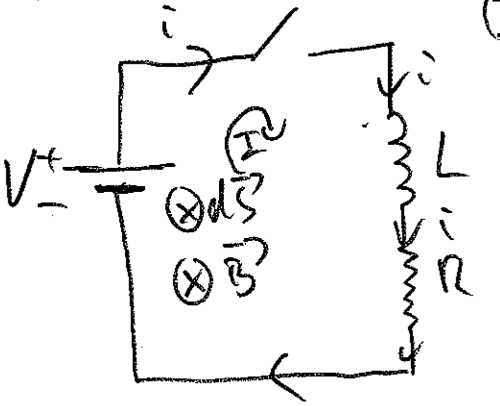
RL circuits

(a) DC series



(I): $-V + Ri = -L\frac{di}{dt}$

Initial condition

$i(0) = 0$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}$$

Homogeneous equation (general solution)

$$\frac{di}{dt} = -\frac{R}{L} i \Rightarrow \frac{di}{i} = -\frac{R}{L}$$

$$\ln\left(\frac{i}{A}\right) = -\frac{Rt}{L}$$

$$\boxed{i(t) = A\exp\left(-\frac{Rt}{L}\right)}$$

Particular solution of inhom. eq.

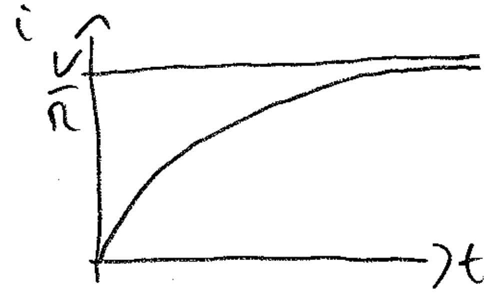
$$i(t) = B = \text{const} \Rightarrow$$

$$\frac{R}{L} B = \frac{V}{L} \Rightarrow B = \frac{V}{R}$$

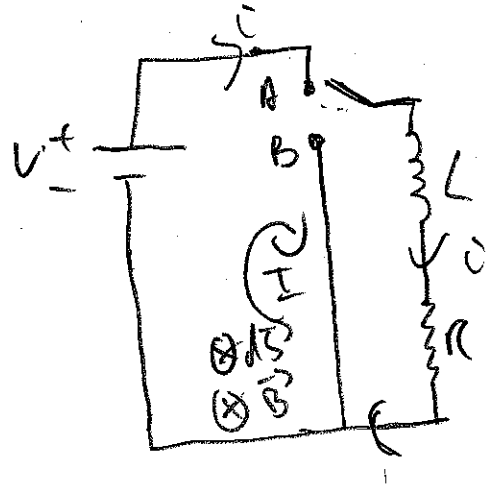
$i(t) = \frac{V}{R} + A\exp\left(-\frac{Rt}{L}\right)$ (general solution of inh. eq.)

$i(0) = \frac{V}{R} + A = 0 \Rightarrow A = -\frac{V}{R}$ (initial condition)

$$i(t) = \frac{V}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]$$



"Decay" of current



Switch for long time in position A

$\Rightarrow t=0: i(0) = i_0 = \frac{V}{R}$

$\Rightarrow t=0:$ switch to pos. B

$$\Rightarrow Ri = -L\frac{di}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = 0 \Rightarrow i(t) = A\exp\left(-\frac{Rt}{L}\right)$$

(homogeneous eq. from above!)

$i(0) = i_0 \Rightarrow A = i_0$

$$i(t) = i_0\exp\left(-\frac{Rt}{L}\right)$$

Energy stored in $\vec{B}$-field

Current runs through $R \Rightarrow$ heat dissipated

$$P = R i^2 = R i_0^2 \exp\left(-\frac{2Rt}{L}\right)$$

Total energy used:

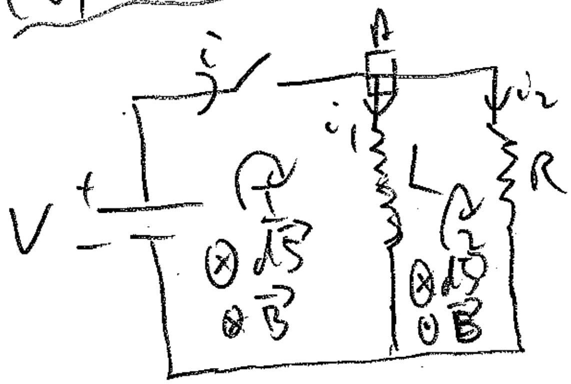
$$E = \int_0^\infty dt\, P(t) = R i_0^2 \int_0^\infty dt \exp\left(-\frac{2Rt}{L}\right)$$

$$E = R i_0^2 \frac{L}{2R}$$

$$\boxed{E = \frac{L}{2} i_0^2}$$

This must have been stored in the $\vec{B}$ field in such the coil when current i_0 ran through!

(b) Parallel circuit



(I) $-V = -L \frac{di_1}{dt}$

(II) $R i_2 = +L \frac{di_1}{dt}$

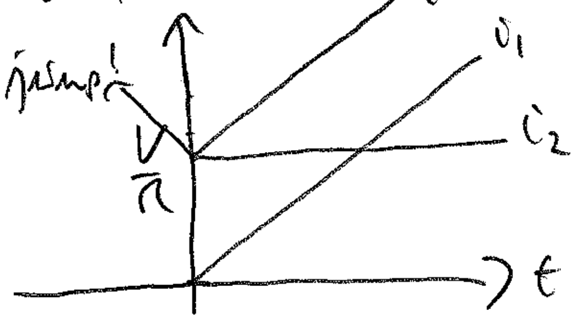
(A) $i = i_1 + i_2$

$i_1(0) = i_2(0) = i_3(0) \stackrel{!}{=} 0$

(I) $\Rightarrow \frac{di_1}{dt} = \frac{V}{L} \Rightarrow i_1(t) = \frac{V}{L} t + A$

$i_1(0) = 0 \Rightarrow \boxed{i_1(t) = \frac{V}{L} t}$

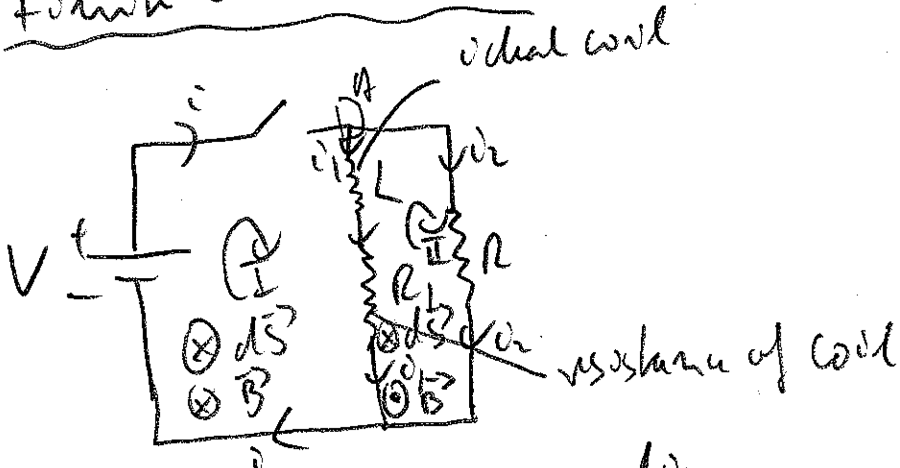
I + II $\quad -V + R i_2 = 0 \Rightarrow i_2 = \frac{V}{R} = \text{const}$



Physically nonsense since i_1 and also i grow to infinity $\Rightarrow$ Reason: neglected resistance of coil (see next section)

jump in i_2 due to negligence of self inductance in loop "big loop"

Finite coil resistance



I $\quad -V + R_1 i_1 = -L \frac{di_1}{dt}$

II $\quad -R_1 i_1 + R i_2 = L \frac{di_1}{dt}$

A $\quad i = i_1 + i_2$

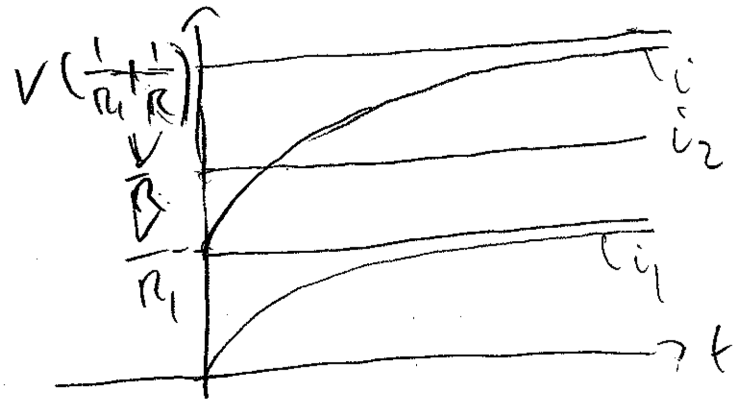
I $\Rightarrow$ same as series circuit:

$$i_1 = \frac{V}{R_1}\left[1 - \exp\left(-\frac{R_1 t}{L}\right)\right]$$

I + II : $-V + R i_2 = 0$

$$i_2 = \frac{V}{R_2}$$

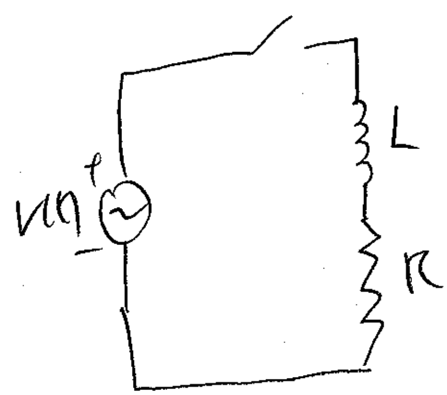
$$i = i_1 + i_2 = V\left(\frac{1}{R_1} + \frac{1}{R}\right) - \frac{V}{R_1}\exp\left(-\frac{R_1 t}{L}\right)$$



For $t \to \infty$ the total current goes to static value (parallel circuit of 2 resistors, R and R_1)

$$i_\infty = V\left(\frac{1}{R_1} + \frac{1}{R}\right)$$

(c) AC Series



Same equations as in (a) but

$$V \to V(t) = V_0 \cos(\omega t)$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}$$

Need to find new sol. for inhom eq.

Ansatz of type of right-hand side

$$i = A_1 \cos(\omega t) + B_1 \sin(\omega t)$$

$$\frac{di}{dt} = -A_1 \omega \sin(\omega t) + B_1 \omega \cos(\omega t)$$

$$\left(-A_1\omega + \frac{R}{L}B_1\right)\sin(\omega t) + \left(\frac{R}{L}A_1 + B_1\omega\right)\cos(\omega t) =$$
$$= \frac{V_0}{L}\cos(\omega t)$$

$$\Rightarrow \frac{R}{L}A_1 + B_1\omega = \frac{V_0}{L}$$
$$-A_1\omega + \frac{R}{L}B_1 = 0 \Rightarrow \boxed{B_1 = \frac{\omega L}{R}A_1}$$

$$\Rightarrow A_1\left(\frac{R}{L} + \frac{\omega^2 L}{R}\right) = \frac{V_0}{L}$$
$$A_1\left(R^2 + \omega^2 L^2\right) = V_0 R$$

$$\boxed{A_1 = \frac{R}{R^2+(\omega L)^2}V_0}$$

$$\boxed{B_1 = \frac{\omega L}{R^2+(\omega L)^2}V_0}$$

Hom solution from case (a) $\Rightarrow$ Full solution

$$i(t) = A\exp\left(-\frac{Rt}{L}\right) + A_1\cos(\omega t) + B_1\sin(\omega t)$$
$$i(0) = A + A_1 \stackrel{!}{=} 0 \Rightarrow A = -A_1$$
$$i(t) = A_1\left[\cos(\omega t) - \exp\left(-\frac{Rt}{L}\right)\right] + B_1\sin(\omega t)$$

$t \gg \frac{L}{R} \Rightarrow$ Stationary state

$$i_\infty(t) = A_1\cos(\omega t) + B_1\sin(\omega t)$$

Amplitude and phase shift

$$i_\infty(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t)$$
$$= A_0 \cos(\omega t + \varphi_0)$$
$$= A_0 \left[\cos(\omega t)\cdot\cos\varphi_0 - \sin(\omega t)\sin\varphi_0\right]$$

$$\Rightarrow \quad A_0 \cos\varphi_0 = A_1$$
$$-A_0 \sin\varphi_0 = B_1$$

$$\Rightarrow A_0^2 = A_1^2 + B_1^2 = \frac{1}{R^2 + (\omega L)^2} V_0^2$$

$$\boxed{A_0 = \frac{1}{\sqrt{R^2 + (\omega L)^2}} V_0}$$

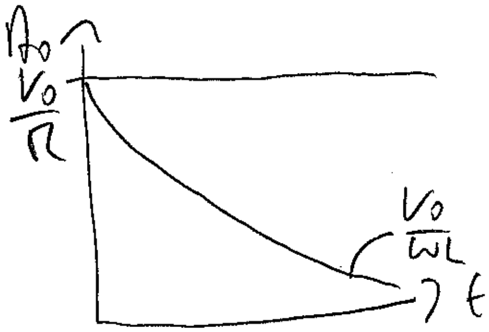
$$\varphi_0 = -\operatorname{sign} B_1 \arccos\left(\frac{A_1}{A_0}\right)$$
$$\varphi_0 = -\arccos\left(\frac{R}{\sqrt{R^2 + (\omega L)^2}}\right)$$

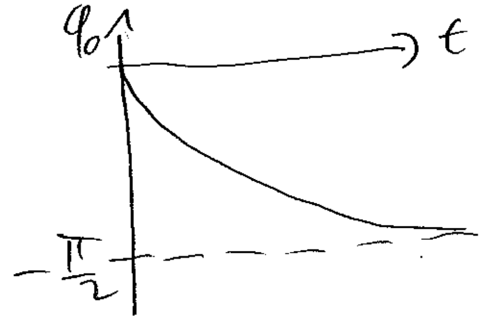
$\omega \to 0$: static limit

$$A_0(\omega \to 0) = \frac{V_0}{R} \quad ; \quad \varphi_0 = 0$$

High-freq. limit

$$A_0 \underset{\omega \to \infty}{\cong} \frac{V_0}{\omega L}$$
$$\varphi_0 \underset{\omega \to \infty}{\cong} -\frac{\pi}{2}$$

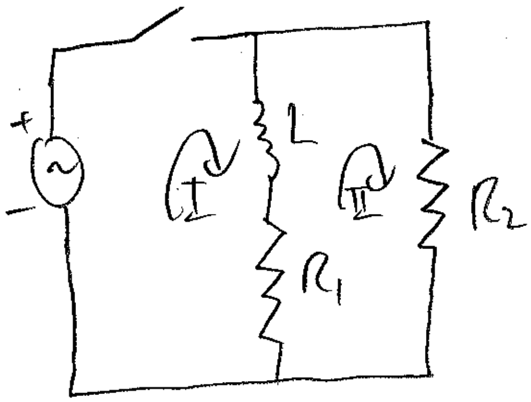




φ_0 always negative
$\Rightarrow$ induced current counteracts change enforced by $V(t)$ (Lenz)

(d) AC parallel

Include R_1 of coil from beginning



(I) has same eq. and solution as discussed in (d)

$$\Rightarrow i_1(t) = A_1\left[\cos(\omega t) - \exp\left(-\frac{R_1 t}{L}\right)\right] + B_1 \sin(\omega t)$$

with $A_1 = \frac{R_1}{R_1^2+(\omega L)^2} V_0$; $B_1 = \frac{\omega L}{R_1^2+(\omega L)^2} V_0$

Same $A_{0,1\infty}$ and $\varphi_{0,1\infty}$ as first discussed

$$i_2(t) = \frac{V_0}{R} \cos(\omega t)$$

$$i(t) = \left(\frac{1}{R} + \frac{R_1}{R_1^2+(\omega L)^2}\right) V_0 \cos(\omega t) + \frac{\omega L V_0}{R_1^2+(\omega L)^2} \sin(\omega t)$$

$$A_{oi}^2 = \left\{ \left(\frac{1}{R} + \frac{R_1}{R_1^2+(\omega L)^2}\right)^2 + \frac{\omega^2 L^2}{[R_1^2+(\omega L)^2]^2} \right\} V_0^2$$

$$= \frac{V_0^2}{[R_1^2+(\omega L)^2]^2 R^2} \left\{ [R_1^2+(\omega L)^2 + RR_1]^2 + (R\omega L)^2 \right\}$$

$$= \frac{V_0^2}{[R_1^2+(\omega L)^2]^2 R^2} \left\{ [R_1^2+(\omega L)^2]^2 + 2RR_1[R_1^2+(\omega L)^2] + R^2R_1^2 + R^2\omega^2L^2 \right\}$$

$$= \text{—''—} \; [R_1^2+(\omega L)^2]\left\{[R_1^2+(\omega L)^2] + 2RR_1 + R^2\right\}$$

$$= \frac{V_0^2}{R^2[R_1^2+(\omega L)^2]}\left\{(R_1+R)^2+(\omega L)^2\right\}$$

$$\Rightarrow \boxed{A_{oi} = \frac{V_0}{R\sqrt{R_1^2+(\omega L)^2}}\sqrt{(R+R_1)^2+(\omega L)^2}}$$

For φ_0:

$$\frac{1}{A_{oi}}\left(\frac{1}{R} + \frac{R_1}{R_1^2+(\omega L)^2}\right)V_0 = \frac{V_0}{A_{oi}} \frac{R_1^2+RR_1+(\omega L)^2}{R[R_1^2+(\omega L)^2]}$$

$$\Rightarrow \varphi_0 = -\arccos\left(\frac{R_1(R+R_1)+(\omega L)^2}{\sqrt{R_1^2+(\omega L)^2}\,\sqrt{(R+R_1)^2+(\omega L)^2}}\right)$$

For $\omega \to 0$

$$A_{0i} \underset{\omega\to 0}{=} V_0 \frac{R+R_1}{RR_1} = V_0\left(\frac{1}{R}+\frac{1}{R_1}\right)$$

$$\varphi_0 \underset{\omega\to 0}{=} 0$$

For $\omega \to \infty$

$$A_{0i} \underset{\omega\to\infty}{\cong} \frac{V_0}{R}$$ (coil acts like an "infinite resistance")

$$\varphi_0 \underset{\omega\to\infty}{\cong} 0$$