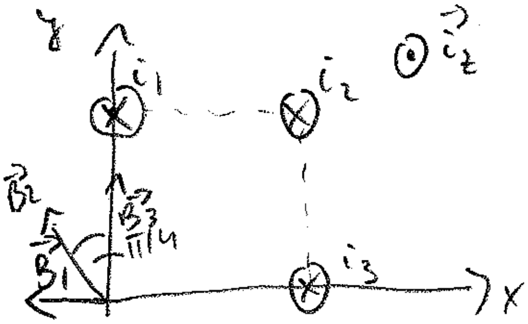
Exam 3: Solutions for final version

(1)

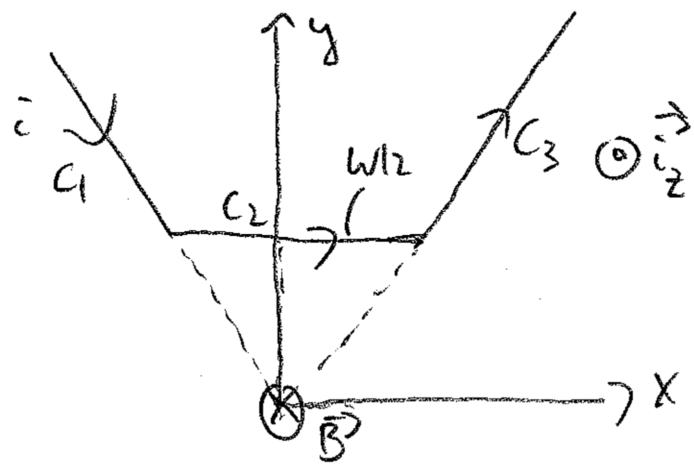


(a) $$\vec{B} = \frac{\mu_0}{2\pi}\left(-\frac{i_1}{w}\vec{i}_x + \frac{i_2}{\sqrt{2}w}\left(\frac{-\vec{i}_x+\vec{i}_y}{\sqrt{2}}\right) + \frac{i_3}{w}\vec{i}_y\right)$$

$$= \frac{\mu_0}{2\pi w}\left[\left(-i_1 - \frac{i_2}{2}\right)\vec{i}_x + \left(\frac{i_2}{2}+i_3\right)\vec{i}_y\right]$$

(b) $$\boxed{\vec{F} = i4\, l\, \vec{i}_z \times \vec{B} = -\frac{\mu_0 l}{2\pi w}\left[\left(\frac{i_2}{2}+i_3\right)\vec{i}_x + \left(i_1+\frac{i_2}{2}\right)\vec{i}_y\right] i4}$$

(2)



$$C = C_1 + C_2 + C_3$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi}\int_C d\vec{r}\,' \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Here we need it for $\vec{r}=0$

$$\vec{B}(\vec{r}=0) = -\frac{\mu_0 i}{4\pi}\int_C d\vec{r}\,' \times \frac{\vec{r}\,'}{|\vec{r}\,'|^3}$$

Along C_1 and C_3 we have

$$d\vec{r}\,' \times \vec{r}\,' = 0$$

Thus only C_2 needs to be considered. Parameterization:

$$C_2: \vec{r}\,' = x\,\vec{i}_x + H\,\vec{i}_y \Rightarrow d\vec{r}\,' = dx\,\vec{i}_x$$

x is running from $-\frac{W}{2} \to +\frac{W}{2}$

$$\vec{B}(\vec{r}=0) = -\frac{\mu_0 i}{4\pi}\int_{-W/2}^{W/2} dx\,\vec{i}_x \times \frac{x\,\vec{i}_x + H\,\vec{i}_y}{\sqrt{x^2+H^2}^{\,3}}$$

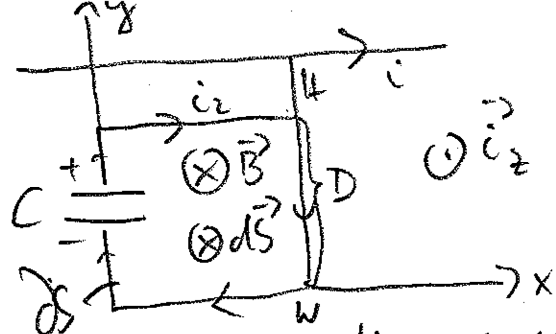
$$= -\frac{\mu_0 i H\,\vec{i}_z}{4\pi}\,2\int_0^{W/2} dx\,\frac{1}{(x^2+H^2)^{3/2}}$$

$$= -\frac{\mu_0 i H\,\vec{i}_z}{2\pi}\left[\frac{x}{H^2\sqrt{x^2+H^2}}\right]_0^{W/2}$$

$$\vec{B}(\vec{r}=0) = -\frac{\mu_0 i W}{4\pi H\sqrt{\frac{W^2}{4}+H^2}}\,\vec{i}_t$$

$$\vec{B}(\vec{r}=0) = -\frac{\mu_0 i W}{2\pi H\sqrt{W^2+4H^2}}\,\mathbf{i}_z$$

(3)



$i = i_0 \cos(\Omega t)$

$\Omega = \omega$ to distinguish from w

The $\vec{B}$ field from the current i in the plane of the circuit is given by

$$\vec{B} = -\vec{i}_z \frac{\mu_0 i_0 \cos(\Omega t)}{2\pi(H+D-y)}$$

Faradays law (signs indicated in figure)

$$\Phi_{\vec{B}} = \int_0^W dx \int_0^D dy \, (-\vec{i}_z)\,\vec{B}$$

$$= \int_0^W dx \int_0^D dy \, \frac{\mu_0 i_0 \cos(\Omega t)}{2\pi(H+D-y)}$$

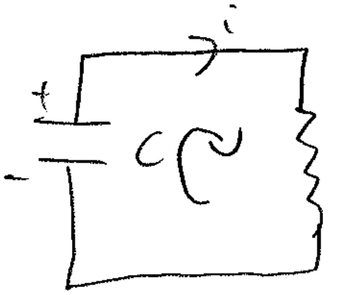
$$= \frac{\mu_0 i_0 \cos(\Omega t)}{2\pi} W \Big[-\ln(H+D-y)\Big]_{y=0}^{D}$$

$$\Phi_{\vec{B}} = \frac{\mu_0 i_0 \cos(\Omega t)}{2\pi} W \ln\left(\frac{H+D}{H}\right)$$

$$\oint_{\partial S} d\vec{r}\cdot\vec{E} = -\frac{Q}{C} = -\frac{d\Phi_{\vec{B}}}{dt} = \frac{\mu_0 i_0 W \Omega \sin(\Omega t)}{2\pi} \ln\left(\frac{H+D}{H}\right)$$

$$\boxed{Q(t) = -\frac{\mu_0 i_0 W \Omega C}{2\pi} \ln\left(\frac{H+D}{H}\right) \sin(\Omega t)}$$

(4) (a) No self-inductance



Faraday's law with $\Phi_{\vec{B}} \equiv 0$

$$\oint \vec{E} \cdot d\vec{r} = 0$$

$$-\frac{Q}{C} + Ri = 0$$

With the direction of the current conducted, we have

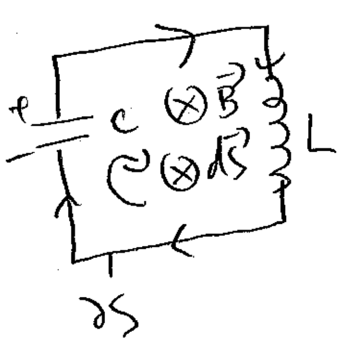
$$i = -\dot{Q} \left(= -\frac{dQ}{dt}\right)$$

$$\dot{Q} = -\frac{Q}{RC}$$

$\Rightarrow$ The solution with $Q(t=0) = Q_0$ is

$$\boxed{Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right)}$$

(b)



$$\oint d\vec{r} \cdot \vec{E} = -\frac{Q}{C} = -\frac{d}{dt}(+Li)$$

$$i = -\frac{dQ}{dt} = -\dot{Q}$$

$$\ddot{Q} = -\frac{1}{LC} Q$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{with } \omega = \sqrt{\frac{1}{LC}}$$

$$i(t) = -\dot{Q}(t) = A\omega\sin(\omega t) - B\omega\cos(\omega t)$$

Initial conditions

$Q(t=0) = Q_0 \Rightarrow A = Q_0$; $i(t=0) = 0 \Rightarrow B = 0$

$$\boxed{\begin{aligned} Q(t) &= Q_0 \cos(\omega t) \\ i(t) &= Q_0\omega\sin(\omega t) \end{aligned} \; ; \; \omega = \sqrt{\frac{1}{LC}}}$$