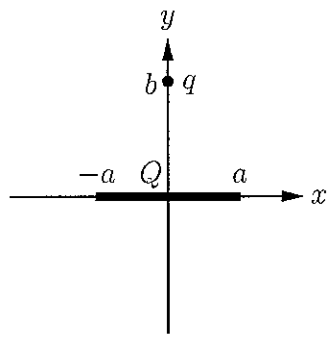
Final Exam Spring 2008
Grading Key

Name: ....

1. (25 points) A charge Q is uniformly spread along the x-axis (from $x = -a$ to $x = a$).



(a) Determine the electric field, $\vec{E}(x, y)$ of this charge distribution.

$$dQ = \frac{Q}{2a}\, dx' \quad 5$$

$$\vec{E} = \frac{1}{8\pi\varepsilon_0 a}\int_{-a}^{a} dx' \, \frac{(x-x')\,\vec{i}_x + y\,\vec{i}_y}{\left[(x-x')^2 + y^2\right]^{3/2}} \quad 5$$

$$= \frac{Q}{8\pi\varepsilon_0 a}\left\{\left[\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{1}{\sqrt{(x+a)^2+y^2}}\right]\vec{i}_x \quad 5\right.$$

$$\left. + \left[\frac{x+a}{\sqrt{(x+a)^2+y^2}} - \frac{x-a}{\sqrt{(x-a)^2+y^2}}\right]\frac{\vec{i}_y}{y}\right\} \quad 5$$

$\boxed{20}$

(b) What is the electric force on a test charge, q, located on the y axis at $y = b$?

$$\vec{F} = q\,\vec{E}\Big|_{\substack{x=0\\ y=b}} = \frac{Q}{4\pi\varepsilon_0}\,\frac{q}{b\sqrt{a^2+b^2}}\,\vec{i}_y$$

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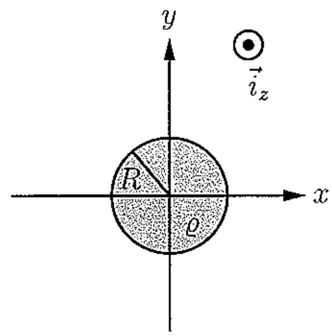
2. (25 points) A sphere of radius, R, is charged with a charge distribution such that the charge density $\varrho(r) = \alpha r$, where $\alpha =$ const.



(a) Determine the electric field, $\vec{E}$, everywhere.

Gauss's Law with sphere of radius r ; $\vec{E} = E_r \vec{i}_r$

$$\oint_{S_r} d\vec{S}\cdot\vec{E} = 4\pi r^2 E_r = \frac{Q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\begin{cases}\pi\alpha r^4 & \text{for } r<R\\ \pi\alpha R^4 & \text{for } r\geq R\end{cases}$$

5

$$\Rightarrow E_r = \begin{cases}\dfrac{\alpha r^2}{4\varepsilon_0} & \text{for } r<R \quad 5\\ \dfrac{\alpha R^4}{4r^2\varepsilon_0} & \text{for } r\geq R \quad 5\end{cases}$$

$\boxed{15}$

(b) Calculate the potential difference between the center of the sphere and a point outside (at a distance $r > R$ from the center).

$$E_r = -\frac{dV}{dr} \Rightarrow V(r) = \begin{cases}\dfrac{-\alpha r^3}{12\varepsilon_0} + C & \text{for } r<R\\ \dfrac{\alpha R^4}{4r\varepsilon_0} & \text{for } r>R\end{cases}$$

$$V(R) = C - \frac{\alpha R^3}{12\varepsilon_0} = \frac{\alpha R^3}{4\varepsilon_0} \Rightarrow C = \frac{\alpha R^3}{3\varepsilon_0}$$

$$V(r) = \begin{cases}\dfrac{\alpha}{3\varepsilon_0}\left(R^3 - \dfrac{r^3}{4}\right) ; & r<R\\ \dfrac{\alpha R^4}{4\varepsilon_0 r} & \text{for } r\geq R\end{cases}$$

$$V(0) - V(r) = \frac{\alpha}{3\varepsilon_0}R^3 - \frac{\alpha R^4}{4\varepsilon_0 r}$$

$\boxed{10}$

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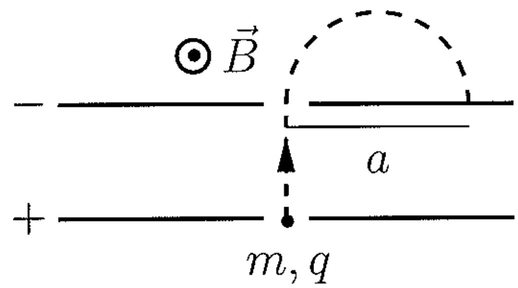
3. (25 points) Two parallel very large conducting plates are connected to a battery with voltage, V, for a long time. Both have a hole in the middle, and at $t = 0$ a charged particle with mass m, and charge, $q > 0$, enters the hole of the lower plate with negligible velocity. Outside of the plates is a homogeneous magnetic field. Gravity can be neglected.



(a) What is the particle's velocity when it leaves the capacitor at the upper plate?

5

(a) $\frac{m}{2} v^2 + qV = \text{const} \Rightarrow qV = \frac{m}{2} v^2$

$$v = \sqrt{\frac{2qV}{m}}$$ 5

$\boxed{10}$

(b) Determine the distance, a, of the point where the particle hits the upper plate again.

$$m \frac{v^2}{R} = qvB \Rightarrow R = \frac{mv}{qB}$$ 5 5

$\boxed{15}$

$$x_{end} = 2R = \frac{2m}{qB} v = \frac{2m}{qB}\sqrt{\frac{2qV}{m}} = \frac{2}{B}\sqrt{\frac{2mV}{q}}$$

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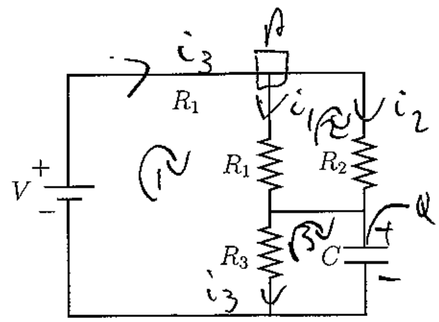
4. (25 points) The circuit is hooked up to the battery as shown for a very long time.
(a) What are the currents through each resistor?



$$\mathrm{I}: -V + R_1 i_1 + R_3 i_3 = 0$$

$$\mathrm{II}: -R_1 i_1 + R_2 i_2 = 0$$

$$\mathrm{III}: -R_3 i_3 + \frac{Q}{C} = 0 \quad 15$$

$$A: \quad i_1 + i_2 = i_3$$

$\boxed{20}$

$$R_1 i_1 + R_3 (i_1 + i_2) = V \; ; \quad i_2 = \frac{R_1}{R_2} i_1$$

$$\left(R_1 + R_3 + \frac{R_1 R_3}{R_2}\right) i_1 = V \Rightarrow i_1 = \frac{R_2 V}{(R_1+R_3)R_2 + R_1 R_3} \quad 5$$

$$i_2 = \frac{R_1 V}{(R_1+R_3)R_2 + R_1 R_3} \; ; \quad i_3 = \frac{(R_1+R_2)V}{(R_1+R_3)R_2 + R_1 R_3}$$

(b) What is the voltage across the capacitor? What is the charge at its positively charged plate?

$$\frac{Q}{C} = V_C = R_3 i_3 = \frac{R_3 (R_1 + R_2) V}{(R_1+R_3)R_2 + R_1 R_3}$$

$\boxed{5}$

Hint: Label all currents and the signs of charges at the capacitor in the circuit diagram!

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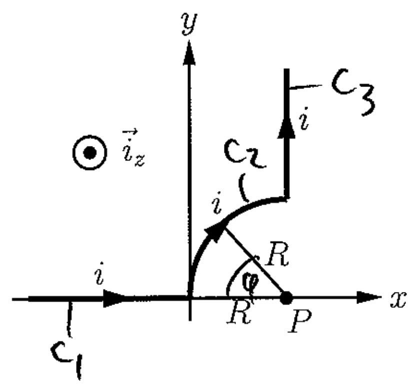
5. (25 points) A thin wire in the xy plane of a Cartesian coordinate system is shaped as shown in the figure: an infinitely long piece lies along the x axis from $x \to -\infty$ to $x = 0$, then a piece is shaped as a quarter of a circle of radius, R. Finally, another infinitely long piece is placed at $x = R$ parallel to the y axis from $y = R$ to $y \to \infty$.



Calculate the magnetic field, $\vec{B}$, at the point P, located at $x = R$, $y = z = 0$, which is the center of the quarter circle!

$$\vec{B}_P = \frac{\mu_0 i}{4\pi} \int_C d\vec{r}\,' \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

$C_1 + C_3$: no contribution since $d\vec{r}\,' \parallel \vec{r} - \vec{r}\,'$ (5)

C_2: $\vec{r}\,' = (R - R\cos\varphi)\vec{i}_x + R\sin\varphi\,\vec{i}_y$; $\vec{r} = R\vec{i}_x$ (5)

$d\vec{r}\,' = d\varphi\,[R\sin\varphi\,\vec{i}_x + R\cos\varphi\,\vec{i}_y]$ (5)

$d\vec{r}\,' \times (\vec{r} - \vec{r}\,') = d\varphi\,[R\sin\varphi\,\vec{i}_x + R\cos\varphi\,\vec{i}_y] \times$ (5)

$[R\cos\varphi\,\vec{i}_x - R\sin\varphi\,\vec{i}_y]$

$= d\varphi\, R^2 (-\vec{i}_z)$

$$\vec{B}_P = -\frac{\mu_0 i}{4\pi} R^2 \vec{i}_z \int_0^{\pi/2} \frac{d\varphi}{R^3} = -\frac{\mu_0 i}{8R}\vec{i}_z$$ (5)

$\boxed{25}$

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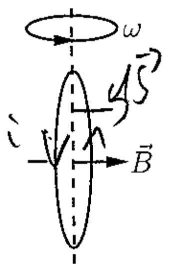
6. (25 points) A circular wire of radius, b, of cross-sectional area, A, is made of a material with resistivity ρ. For a very long time, it rotates around one of its diameters with constant angular velocity, ω. A homogeneous magnetic field, $\vec{B}$, is pointing perpendicular to the rotation axis.



(a) Calculate the current induced in the wire.

$$R\, i = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\pi b^2 B \cos(\omega t)\right] \quad (10)$$

$$i = +\frac{\pi b^2 B \omega}{R} \sin(\omega t) \quad (5)$$

$$R = \frac{\rho l}{A} = \frac{2\pi b \rho}{A} \quad (5)$$

$\boxed{20}$

(b) How much energy is used during one period of rotation $T = 2\pi/\omega$?

$$P(t) = R[i(t)]^2 = \frac{\pi^2 b^4 B^2 \omega^2}{R^2} \cdot [\sin(\omega t)]^2 \quad (3)$$

$$E_{1\,rev} = \int_0^T dt\, P(t) = \frac{\pi^3 b^4 B^2 \omega}{R^2} \quad (2)$$

$\boxed{5}$

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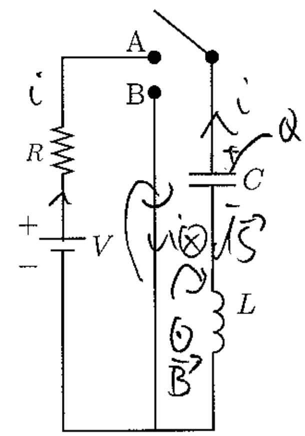
7. (25 points) (a) In the circuit shown in the figure, the switch has been in position A for a long time.



(a) What is the charge at the upper plate of the capacitor?

$$-V + \frac{Q_0}{C} = 0 \Rightarrow Q_0 = CV \quad \boxed{5}$$

(b) At $t = 0$ the switch is set into position B. Calculate the charge, $Q(t)$, at the upper plate of the capacitor as a function of time!

$$\frac{Q}{C} = +L\frac{di}{dt} \; ; \; i = -\dot{Q} \Rightarrow \ddot{Q} = -\frac{1}{LC}Q \quad 5$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t) \; ; \; \omega = \frac{1}{\sqrt{LC}} \quad 5$$

$$i(t) = -\dot{Q} = A\omega\sin(\omega t) - B\omega\cos(\omega t)$$

$$Q(0) = Q_0 \; ; \; i(0) = 0 \Rightarrow A = Q_0 \; ; \; B = 0 \quad 5$$

$$Q(t) = Q_0\cos(\omega t)$$

$\boxed{15}$

(c) What is the current, $i(t)$, through the coil, as a function of time (for $t > 0$)?

$$i(t) = A\omega\sin(\omega t)$$

$\boxed{5}$

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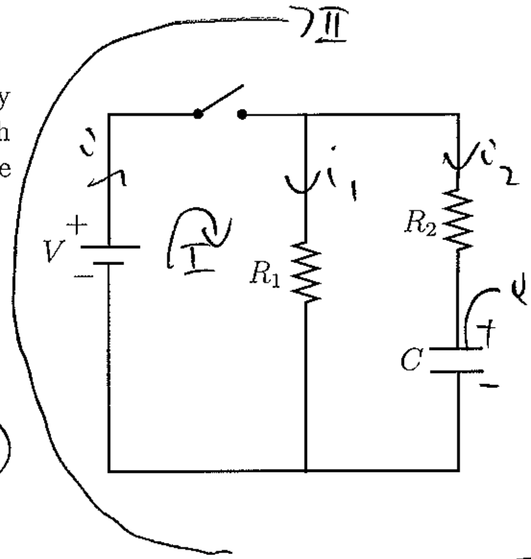
8.(25 points) (a) Suppose the switch has been open for a very long time. At $t = 0$ it is closed. Calculate the currents through the resistors as a function of time! All self-inductances can be neglected.



$\underline{\mathrm{I}}: -V + R_1 i_1 = 0 \Rightarrow i_1 = \frac{V}{R_1}$ 5

i_1 jumps from 0 to steady-state value

$\underline{\mathrm{II}}: -V + R_2 i_2 + \frac{Q}{C} = 0$; $i_2 = \dot{Q}$ 5

$\Rightarrow \dot{Q} + \frac{Q}{R_2 C} - \frac{V}{R_2} = 0$ 5

$Q(t) = A \exp(-\frac{t}{R_2 C}) + CV$; $Q(0) = 0 \Rightarrow A = -CV$

$\boxed{20}$

$Q(t) = CV[1 - \exp(-\frac{t}{R_2 C})]$ 5

(b) Show that after a long time ($t \to \infty$) the currents reach the values to be expected from a steady-state situation.

i_1 jumps immediately to the steady-state value

$\boxed{5}$

$i_2(t) = \frac{V}{R_2} \exp(-\frac{t}{R_2 C})$

The uncharged capacitor is like a short-circuit in the very first moment. Then the current decays to 0, which is the steady-state value.