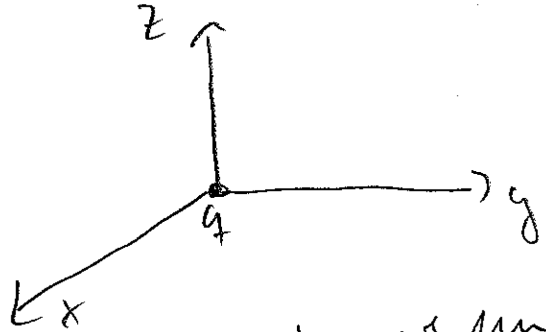
5. Applications of Gauss's Law (Feb/14)

Often we can use Gauss's law in the integral form we have discussed in the last chapter, to calculate the electric field. This is the case, if symmetries of the problem allow an ansatz for the field. We shall discuss some examples

(a) A single point charge

We have derived Gauss's law from Coulomb's law. Now we'll see that we can derive also Coulomb's from Gauss's law. Let the single point charge be at the origin of our coordinate system



Due to spherical symmetry, we must have

$$\vec{E}(\vec{r}) = E_r(r)\,\vec{e}_r,$$ (because $\vec{E} = -\text{grad}\,V$ with $V = V(r)$ in spherical coordinates!)

and we can find $E(r)$ by applying Gauss's Law to a sphere

$$\int_{S_R} d\vec{S}\cdot\vec{E} = \frac{q}{\varepsilon_0}$$

$d\vec{S} = R^2 \sin\vartheta \, d\vartheta \, d\varphi \, \vec{e}_r$ in our standard spherical coordinates, and thus

$$\frac{q}{\varepsilon_0} = \int_{S_R} d\vec{S}\cdot\vec{E} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, R^2 \sin\vartheta \, E_r(R)$$

$$= 4\pi R^2 \, E_r(R)$$

and thus, again setting $R = r$

$$E_r(r) = \frac{q}{4\pi\varepsilon_0 r^2}$$

or
$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{i}_r = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

which is again Coulomb's Law as it must be.

<u>(b) Uniformly charged ball</u>

Take a ball of radius R with uniform charge. Then again due to spherical symmetry we must have

$$\vec{E} = E_r(r)\,\vec{i}_r$$

Again we use Coulomb's laws as in (a). If $r > R$ then we get again

$$E_r(r) = \frac{q}{4\pi\varepsilon_0 r^2} \; !$$

o.k., Coulomb's law, because all the charge, q, sits inside the surface. But now, if $r < R$, only the charge inside the sphere with radius r contributes, but this is

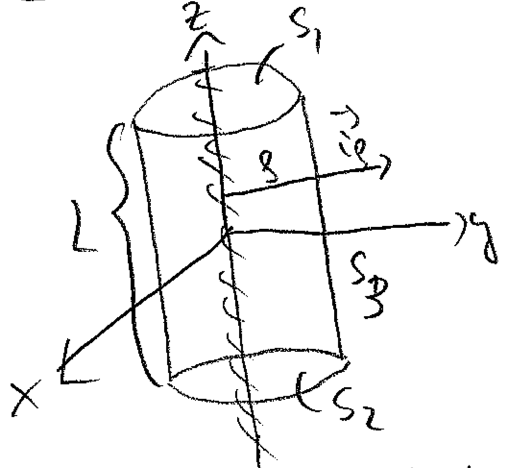
$$Q_{S_r} = \rho\, V_r = \frac{q}{\frac{4\pi}{3}R^3}\,\frac{4\pi}{3}r^3 = \frac{Q}{R^3}r^3$$

The LHS of Gauss's law is again

$$4\pi r^2 E_r = \frac{Q_{S_r}}{\varepsilon_0} = \frac{q}{\varepsilon_0 R^3}r^3$$

and thus
$$E_r = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 R^3}\, r & \text{for } r < R \\[2ex] \dfrac{q}{4\pi\varepsilon_0 r} & \text{for } r \geq R \end{cases}$$

(c) Wire of uniformly distributed charge



Symmetry demands that in cylinder coordinates, ϱ, φ, z

$\vec{E} = E_\varrho \vec{i}_\varrho$ (because $V = V(\varrho)$ and grad $V = \frac{\partial V}{\partial \varrho} \vec{i}_\varrho$).

As a gaussian surface we consider a cylinder of length L. The electric flux through the top and bottom caps are 0 since there $d\vec{S} \sim \vec{i}_z$ and $\vec{i}_z \cdot \vec{i}_\varrho = 0$ for all ϱ, φ, z.

The remaining part is the curved surface, S_3 and for that

$$d\vec{S}_3 = \varrho \, d\varphi \, dz \, \vec{i}_\varrho \text{ with } \varphi \in (0, 2\pi); \; z \in [-\tfrac{L}{2}, \tfrac{L}{2}]$$

Thus

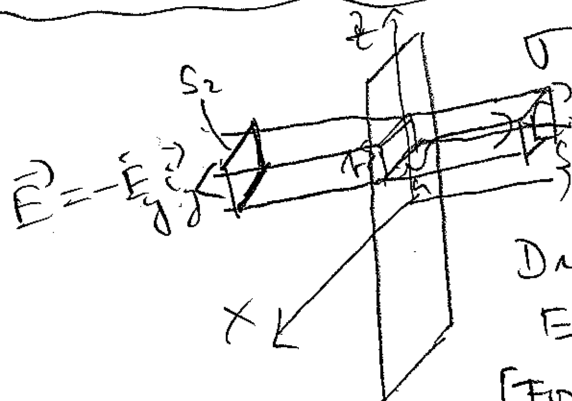
$$\int_{S_3} \vec{E} \cdot d\vec{S}_3 = \int_0^{2\pi} d\varphi \int_{-L/2}^{L/2} dz \, \varrho E_\varrho(\varrho) = 2\pi L \varrho \, E_\varrho(\varrho)$$

$$\overset{!}{=} \frac{Q_V}{\varepsilon_0} = \frac{L\lambda}{\varepsilon_0}$$

$$\Rightarrow E_\varrho(\varrho) = \frac{\lambda}{2\pi\varepsilon_0 \varrho}$$

$$\Rightarrow \vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0 \varrho} \vec{i}_\varrho = \frac{\lambda}{2\pi\varepsilon_0} \frac{x\vec{i}_x + y\vec{i}_y}{x^2 + y^2}$$ $\Rightarrow$ Quiz II!

(d)
Uniformly charged infinite plane



σ : charge per area

symmetry!

Due to mirror symmetry we have
$E_y(y) = -E_y(-y)$.
[For the potential: $V(y) = V(-y)$]

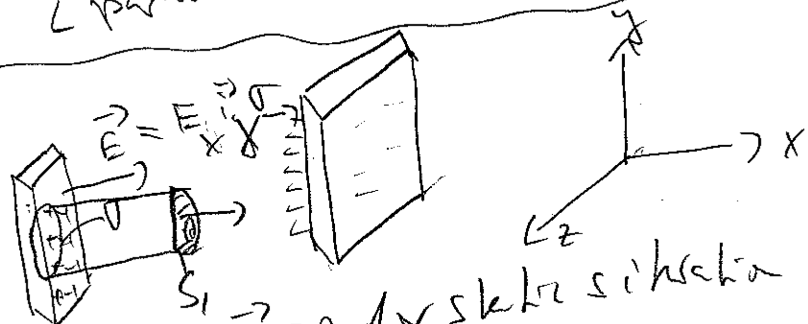
Surface as shown:

$$\frac{Q_V}{\varepsilon_0} = \frac{\sigma a b}{\varepsilon_0} = \int_{S_1+S_2} d\vec{S} \cdot \vec{E} = 2 E_y a b$$

$$\Rightarrow E_y = \frac{\sigma}{2\varepsilon_0}$$

Note: $\vec{E} \cdot \vec{n}$ makes jump of size $\frac{\sigma}{\varepsilon_0}$ as it should be.

(e) 2 parallel conducting plates (infinite)



In the plates $\vec{E} = 0$ for static situation $\Rightarrow$
for cylinder as drawn

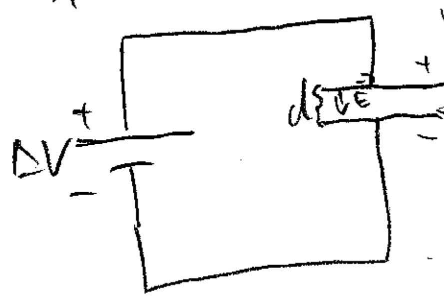
$$\frac{\sigma A}{\varepsilon_0} = \int_{S_1} d\vec{S} \cdot \vec{E} = E_x(x) A$$

$$\Rightarrow E_x(x) = \frac{\sigma}{\varepsilon_0} = \text{const.} \quad \text{(between the plates)}$$

$\vec{E} = 0$ outside the plates.

Chapter VI Capacitors (Feb 11/9)

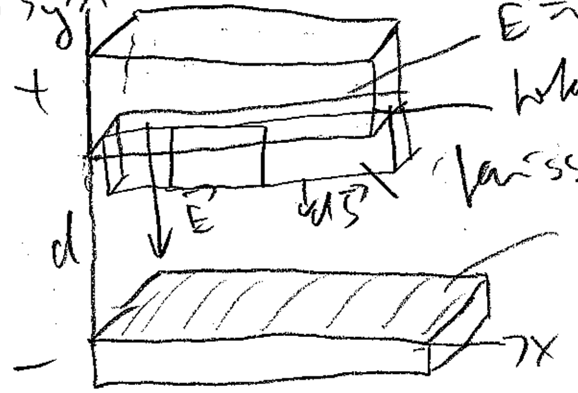
Ideal battery: Device which delivers a given electric-potential difference. The "+ terminal" has a electric potential larger by the amount ΔV compared to the "– terminal." An ideal conductor is always an equipotential line or surface. Now we connect the ideal battery with a capacity, i.e., two conducting plates, separated by vacuum:



If we make the extension of the area of the plates $\gg d$, we can assume an infinite area, and we get $\vec{E}$ as in the previous chapter as

$$\vec{E} = E_y(y)\,\vec{i}_y$$

Using a cylinder with one plate, say the upper, in such Gauss's law works



$$\vec{E} = E_y(y)\,\vec{i}_y$$

area of plate

$$\Rightarrow \oint \vec{E}\cdot d\vec{S} = -E_y(y)\,A = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E_y(y) = -\frac{Q}{A\varepsilon_0} = \text{const.}$$

Potential difference [integrate along path from plate at $y=0$ to plate at $y=d$ in y-direction

$$-\int_{y=0}^{y=d} d\vec{r}\,\vec{E} = \frac{Qd}{A\varepsilon_0} = \Delta V$$

(To get a positive ΔV, integrate from "–" to "+").

The quantity

$$C = \frac{Q}{\Delta V} = \frac{A\varepsilon_0}{d}$$

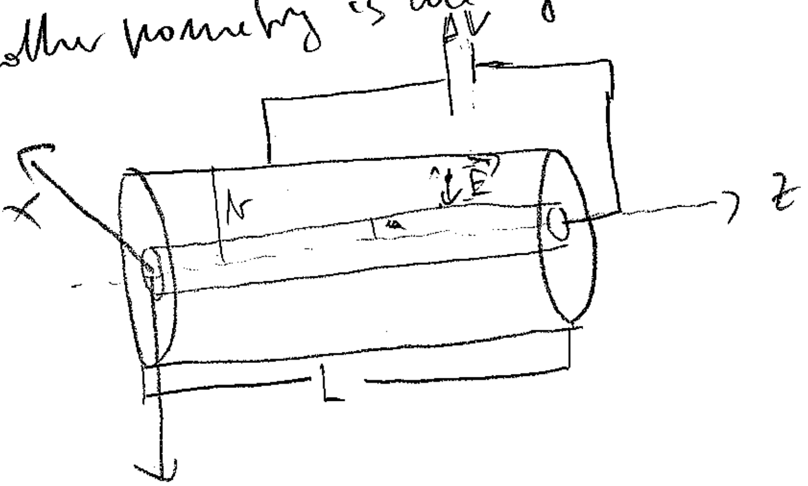
is only dependent on the geometry of the capacitor, not on the voltage, ΔV, or charge, Q. It's called capacitance and gives the charge Q on the upper plate, if a voltage ΔV is applied to the capacitor. Then we have

$$Q = CU$$

The unit obviously is

$$[C] = \frac{\text{Coulombs}}{\text{Volts}} = F$$ (Farad after the great experimental physicist Michael Faraday)

Another geometry is the cylinder capacitor



If $L \gg b$ we can neglect edge effects at the ends, and we have the same situation as in the previous chapter

For $a < s < b$ we have

$$\vec{E} = E_s(s)\,\hat{e}_s \text{ with } E_s(s) = \frac{-\lambda}{2\pi\varepsilon_0 s} = -\frac{Q}{2\pi\varepsilon_0 L}$$

The voltage is given by the potential difference between the conductors, most easily found by integrating radially between the plates.

$$\Delta V = -\int_{s=a}^{b} \vec{E}\,d\vec{r}$$

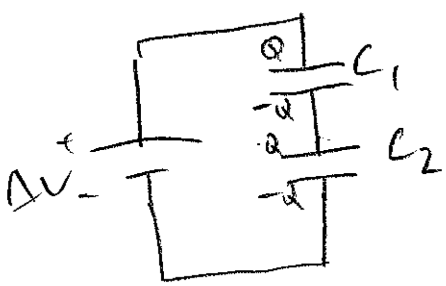
$$= +\int_a^b ds\,\frac{Q}{2\pi\varepsilon_0 L s} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

The capacitance thus is

$$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

which again is only dependent on the geometry not on Q (or ΔV).

Capacitors in Series



$\Delta V = \Delta V_1 + \Delta V_2$ (consider an integration path along the connection between the capacitors from the upper plate of C_1 to the lower plate of C_2)

On the other hand

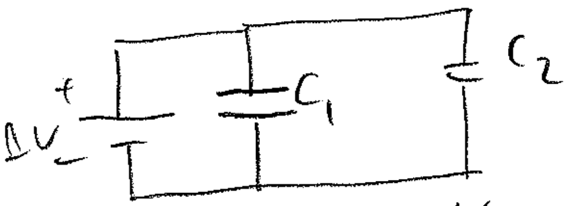
$$\Delta V_1 = \frac{Q}{C_1}\ ;\quad \Delta V_2 = \frac{Q}{C_2}$$

$$\Rightarrow \Delta V = \Delta V_1 + \Delta V_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

The effect is the same as that of a single capacitor with capacity C, given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow \boxed{C = \frac{C_1 C_2}{C_1 + C_2}}$$

Capacitors in parallel



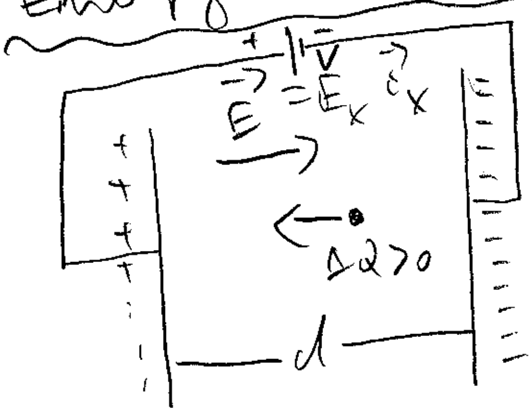
$$\Delta V = \Delta V_1 = \Delta V_2$$

$$Q_{total} = Q_1 + Q_2 = C_1 \Delta V_1 + C_2 \Delta V_2 = (C_1 + C_2)\Delta V$$

Same effect as single capacitor with capacitance

$$C = C_1 + C_2$$

Energy in capacitor



$$E_x = \frac{Q}{\varepsilon_0 A} = \text{const.}$$

$$V = \frac{Q d}{\varepsilon_0 A}$$

$$\Delta W = \Delta Q \, E_x d = \Delta Q \, V$$

For infinitesimal charge ΔQ we find for total work done

$$W = \int_0^Q dQ \, V = \int_0^Q dQ \, \frac{Q}{C} = \frac{Q^2}{2C} = \frac{C^2 V^2}{2C} = \frac{C}{2} V^2$$

This is the energy contained in the capacitor, if a voltage ΔV is applied:

$$\boxed{W = \frac{C}{2} V^2}$$

In a metal, the electrons move freely. This makes metals good conductors. In a battery, by chemical processes, we won't discuss in further detail, we have a more or less stable potential difference between the terminals.

If we now connect the battery with ideally conducting wires, this potential difference will be applied to the capacitor.

Thus charges must have moved such that the upper plate becomes a positive charge $C \cdot \Delta V$, and the lower plate a charge $-C\Delta V$. Really the electrons flow from the upper plate to the + terminal of the battery. The chemistry works to keep up the potential difference. So negative charge is pushed to the lower plate of the capacitor.

So for a certain time charges must have been moving, and this is called an electric current.

Steady-State Currents and Real Wires

Now we look at currents more quantitatively. We define the current through a wire as the amount of charge ΔQ which flows through it in a time interval Δt:

$$i = \frac{\Delta Q}{\Delta t}.$$

or the momentaneous current

$$i = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}.$$

The unit of current is

$$[i] = \frac{C}{s} = A$$ (Ampère named after an important French physicist)

A real wire is not ideal (as the name "ideal" suggests). But there is a certain resistance against the flow of charges (electrons) since those hit each other and the lattice of atoms the metal consists of.

The German physicist Ohm (in fact he was a high-school teacher) found that through a wire connecting an ideal battery of voltage V, i is given by

$$i = \frac{V}{R} \quad \text{with a constant } R,$$

the resistance of the wire. Its unit is called Ohm abbreviated with Ω:

$$[R] = \frac{[V]}{[i]} = \frac{V}{A} = \frac{\text{Volts}}{\text{Ampère}} = \text{Ohm} = \Omega$$

It is found that

$$R = \rho \frac{l}{A} \quad \left(\begin{array}{l} l: \text{length of wire} \\ A: \text{cross sectional area} \end{array} \right)$$

where ρ is the specific resistance, and this depends on the material (and the temperature) only.

Another quantity is the conductivity of a material

$$\sigma = \frac{1}{\rho}$$

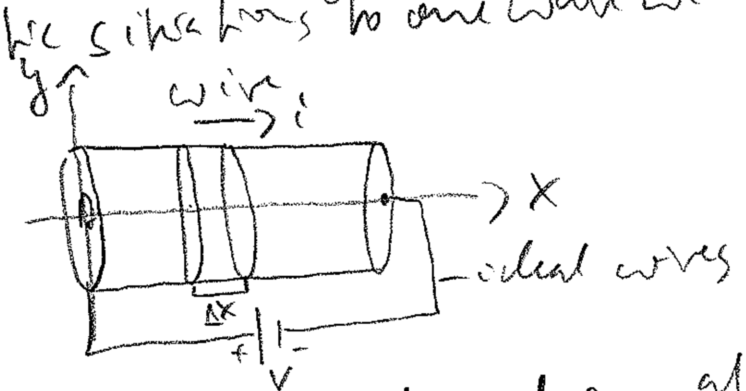
The unit is

$$[\sigma] = \frac{1}{[\rho]} = \frac{1}{\Omega\, m} = \frac{1\,℧}{m}$$

$℧$ = Mho (Ohm spelled backwards).

Note: In Europe the unit is also $1℧ = 1$ Siemens named after the engineer and inventor Siemens.

Now we want get a more local picture of currents. It should in fact be a vector. Thus we try to figure out the relation between the current and the electric field in a real wire. Note that strictly speaking this is a new concept because we apply our ideas about the electric field from static situations to one where we have a steady current!



Now suppose two surfaces drawn at x and $x + \Delta x$.
According to Ohm's law we have

$$\Delta V = V(x+\Delta x) - V(x) = -\frac{i\rho}{A}$$ (the − sign is due to the fact that by definition the potential is higher at the + terminal!)

$$\text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \frac{\partial V}{\partial x} = -\frac{i\rho}{A} = -E_x$$

$$\text{or} \quad E_x = \sigma \frac{i}{A}$$ ($\sigma = \frac{1}{\rho}$ = conductivity of the material).

Local form of Ohm's law

We define the current through an area A by

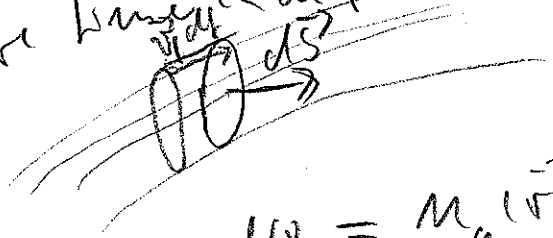
$$i_A = \int_A d\vec{S} \cdot \vec{j}.$$

where $d\vec{S}\cdot\vec{j}$ is the number of charges which flow through the surface element $d\vec{S}$ in unit of time. This is very intuitive:
Now we can repeat our idea of fluid flow with the charge carried by the particles (in normal metals these are the electrons).
Note that i (current) is a flux, i.e. a scalar and only properly defined together with the area and the orientation of $d\vec{S}$ which is arbitrary!

If there is a charge density of moving charges

$n_q(\vec{r})$,

and the velocity of the particles at position $\vec{r}$ is $v(\vec{r})$.
(We consider steady states only, i.e. $n_Q(\vec{r})$ and $\vec{v}(\vec{r})$ are time independent). Then, in a time dt



a charge $dQ = n_q(\vec{r})\,\vec{v}(\vec{r})\,dt\,d\vec{S}$
passes through this surface element. This makes a current

$$di = \frac{dQ}{dt} = n_q(\vec{r})\,\vec{v}(\vec{r})\,d\vec{S}$$

and thus $n_q(\vec{r})\,\vec{v}(\vec{r}) = \vec{j}(\vec{r})$

More precisely we must write

$$\vec{j}(\vec{r}) = n_-(\vec{r})\vec{v}_-(\vec{r}) + n_+(\vec{r})\vec{v}_+(\vec{r})$$

where n_- is the density of negative charges (<0!) and n_+ that of the positive charges. In a metal we have $\vec{v}_+ = 0$. And thus

$$\vec{j}(\vec{r}) = n_-(\vec{r})\,\vec{v}(\vec{r}) \left[= -|n_-(\vec{r})|\,\vec{v}(\vec{r})\right],$$

while

$$n_{total}(\vec{r}) = n_-(\vec{r}) + n_+(\vec{r}) = 0. \, *$$

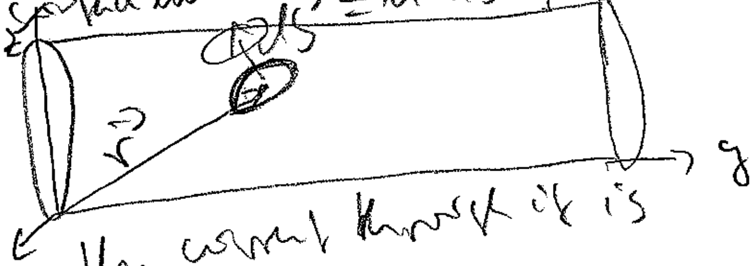
---

*Note that there could be static surface charges, described by a surface-charge density $\sigma_Q(\vec{r})$, distributed on the surface of the conductor.

We note that in the steady state

$$\oint_{\partial V} d\vec{S}\cdot\vec{j}(\vec{r}) = 0$$

because of charge conservation. At any moment as much of charge is flowing out as is flowing into the closed boundary. Now we can make the argument more stringent. Take a small area in the wire, with an arbitrary direction of its surface element $d\vec{S} = \vec{n}\, dS$; $\vec{n}$: unit vector ⊥ to surface



Then the current through it is

$$di = \vec{n}\,\vec{j}\, dS$$

Now take a path along $-\vec{n}$:

$$d\vec{r} = -da\,\vec{n}$$

Then the difference in the electric potential is

$$dV = -d\vec{r}\cdot\vec{E} = da\,\vec{n}\cdot\vec{E} = +R\,di = +R\,\vec{n}\,\vec{j}\,dS$$

$$\Rightarrow \vec{n}\,\vec{E} = +R\,\vec{n}\cdot\vec{j}\,\frac{dS}{da} = \rho\,\vec{n}\cdot\vec{j} = \frac{\vec{n}\cdot\vec{j}}{\sigma},$$

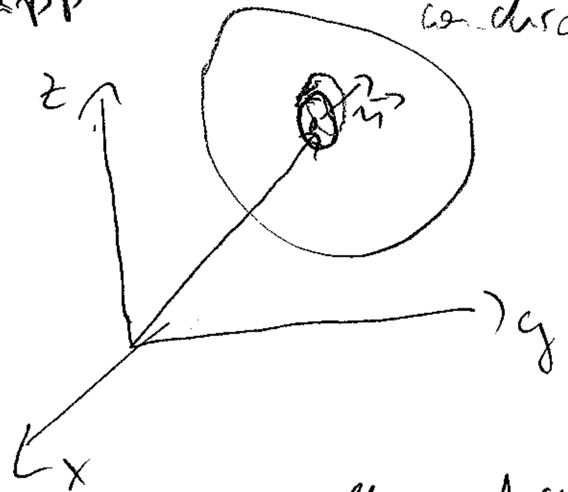
but $\vec{n}$ is an arbitrary unit vector. Setting it $\vec{i}_x$, $\vec{i}_y$ and $\vec{i}_z$ of an arbitrary coordinate system, we finally find again

$$\vec{E} = \rho\,\vec{j} = \frac{\vec{j}}{\sigma} \quad \text{or} \quad \boxed{\vec{j} = \sigma\vec{E}}$$

The direction of the current density is always from the higher to the lower potential! Or to remember "Voltage drops in the direction of the flow of charges, given by the charge density $\vec{j}$". Note that the current is a flux, i.e., a scalar.

Local version of Ohm's law again (02/26/07)

Suppose a current density, $\vec{j}(\vec{r})$, in a conductor



Take a very small surface with normal unit vector $\vec{n}$ and surface dS. Then the current through this surface is

$$i = \vec{j}\,\vec{n}\,dS$$

(Surface vector, $\vec{n}$, defining the current's sign)

Consider a parallel surface

$$S' : \vec{r}_{S'} = \vec{r}_S + \vec{n}\,dl$$

The potential difference along the direction against the

$$\Delta V = -\left[V(\vec{r}_{S'}) - V(\vec{r}_S)\right] = -\left(\frac{\partial V}{\partial x} n_x + \frac{\partial V}{\partial y} n_y + \frac{\partial V}{\partial z} n_z\right) dl = \vec{E}\,\vec{n}\,dl$$

On the other hand that's by definition

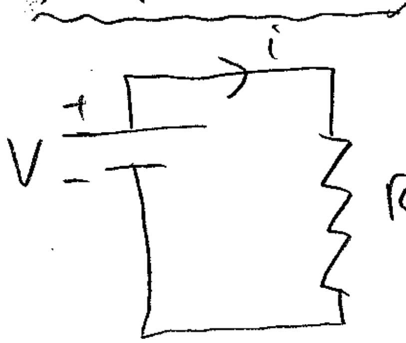
$$R\,i = R\,\vec{j}\cdot\vec{n}\,dS = \rho\,\vec{j}\cdot\vec{n}\,dl = \vec{E}\,\vec{n}\,dl \quad (*)$$

$$\Rightarrow \vec{E} = \rho\,\vec{j}$$

since (*) holds for each $\vec{n}$ (especially for $\vec{n} = \vec{i}_x, \vec{i}_y$, and $\vec{i}_z$!).

Chapter VIII: Joule's law, EMF, and simple circuits

8.1 Joule's law (DC): (DC = "Directed Current" = Steady-state current)



The total energy gained by a charge passing through the wire from one terminal of the battery back to the battery is 0 since we still have electrostatics for DC situations.

On the other hand, along R we have an electric field, given by $\vec{E} = \rho \vec{j}$ yielding a voltage of $V = Ri$. For a charge, dq, passing through the resistor, an amount of energy

$$dU = V\,dq = V i\,dt = R i^2\,dt$$

must be lost to the particle. This happens by converting the kinetic energy of the electrons passing through the wire, by collisions with the lattice, heating up the wire. The power (energy loss per time) is

$$P = \frac{dU}{dt} = Vi = Ri^2 = \frac{V^2}{R} \quad \text{(Joule's Law)}$$

Unit:

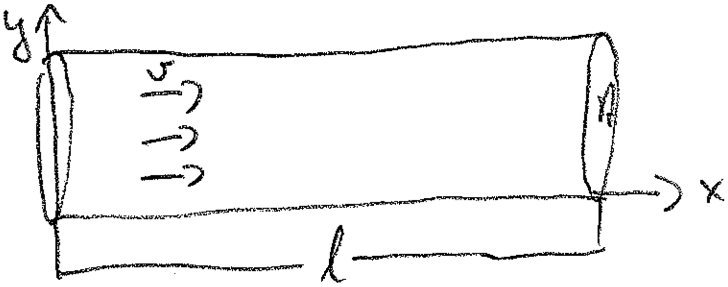
$$[P] = VA = \frac{J}{sec} = W$$ (Watts named after James Watt)

Sign of ΔV was already discussed

8.2 Drift velocity (Feb/28)

We like to determine the velocity of the electrons making the current in a wire.

Let n_- be the charge density of electrons ($n_- < 0$!) 64



Consider the right end of the wire. We have a current density

$$\vec{j} = -e n_- \vec{v} = \frac{i}{A}(-\vec{i}_x) = -e n_- v \, \vec{i}_x$$

$$\Rightarrow v = \frac{i}{e n_- A}$$

In our DC situation, $\vec{j}$ must be constant due to

$$\oint_S \vec{j}\, d\vec{S} = 0 \text{ for all closed surfaces } S.$$

For a given voltage difference at the ends of the wire, we have

$$i = \frac{V}{R} = \frac{V}{\frac{\rho l}{A}} = \frac{VA}{\rho l}$$

$$\Rightarrow v = \frac{V}{e \rho n_- l}$$

For a typical material as copper we have about 1 free electron per atom, making

$$n \simeq 10^{23} \frac{1}{cm^3} = 10^{23} \frac{1}{(10^{-2} m)^2} = \frac{10^{29}}{m^3}$$

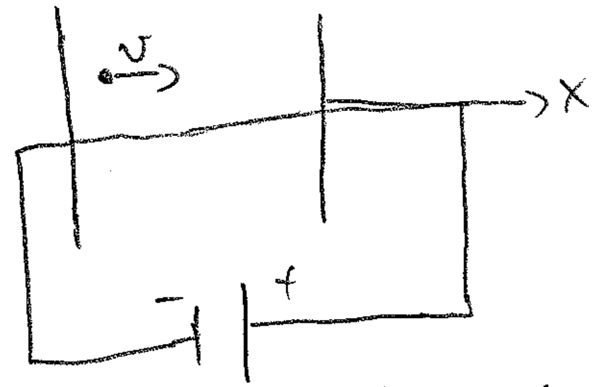
$e = 1.6 \cdot 10^{-19} C$

$\rho \approx 10^{-8} \Omega m$ (Cu)

Take $V = 1V$

$\Rightarrow v \gtrsim 6 \cdot 10^{-3} \frac{m}{s}$ (= 0.01 mph)

Compare this to a free electron accelerated by a potential difference of 1V:



initial: $v_0 = 0, x = 0$;
final: $v = ?, x = l$

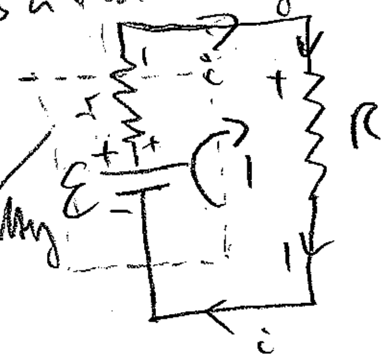
Constant electrical potential from $x=0$ plate. Then

sign because $q_{electron} = -e$

$$E = E_0 = 0$$
$$E = E_{final} = \frac{m}{2} v^2 - eV = 0$$
$$\Rightarrow v = \sqrt{\frac{2eV}{m}} \simeq 6 \cdot 10^5 \frac{m}{s}$$

8.3 Electromotive force

A real battery has a finite resistance, may it be as low as it is.
Thus a real battery can be represented by



Which voltage is on R?
Everywhere in the circuit runs the same current, i. Now we integrate $\vec{E}$ along this circuit, giving the current an arbitrary direction. The equations will always give the right signs. We only must remember that, when integrating "from − to +" of a battery, we get its positive voltage. Then we assign + and − signs at the ends of a resistor such that the currents through them go from + to −. Then the same rules as

for batteries apply for the voltage Ri on the resistor.
In our case thus we have:

$$\mathcal{E} - ir - iR = 0,$$

and it's 0, because for DC situations $\vec{E}$ is still conservative, i.e. the integral over $-\oint \vec{E}\, d\vec{r}$ along an arbitrary closed path vanishes; thus the voltage on the resistor is

$$V_R = iR = \mathcal{E} - ir$$

where $\mathcal{E} = ir + iR = (r+R)i \Rightarrow i = \frac{\mathcal{E}}{r+R}$

and thus

$$V_R = \mathcal{E} - \frac{\mathcal{E}}{r+R} = \mathcal{E}\left(1 - \frac{r}{r+R}\right) = \frac{R}{r+R}\mathcal{E}$$

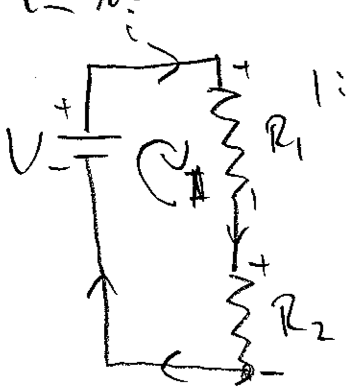
$$V_R = \frac{R}{R+r}\mathcal{E}$$

Thus the voltage given on the battery, $\mathcal{E}$, the socalled "electromotive force", is only on R for $R \to \infty$ (i.e. $R \gg r$).
It can be measured with an volt meter with very large resistance.

8.4 Simple circuits (Mar/02)

(a) Resistors in Series

From now on we use ideal batteries again, always keeping in mind that the battery in reality has a finite resistance.



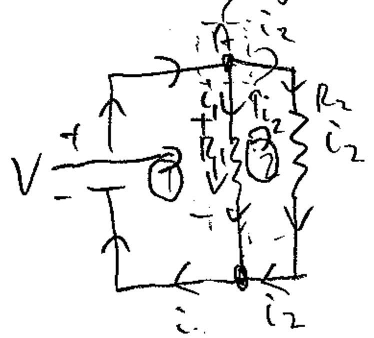
$$1: \quad V - iR_1 - iR_2 = 0$$

$$\Rightarrow V = (R_1 + R_2)\, i$$

Thus $R_{tot} = R_1 + R_2$

(iv) resistors in parallel

box around knot A



Now we have according to our rule

$$1: \; V - i_1 R_1 = 0$$
$$2: \; i_1 R_1 - i_2 R_2 = 0$$

That's not enough to solve for i_1 and i_2 and to get R_{total}. We need in addition the conservation law for charges. For DC's the total flux through a closed surface must vanish:

$$\oint_S d\vec{S} \cdot \vec{j}(\vec{r}) = 0 \; \text{(for all closed surfaces).} \quad (*)$$

Thus we draw a closed box around the branching point A (also called a knot in the circuit). Then the continuity eqn. for (*) yields

$$-i + i_1 + i_2 = 0$$

(note that the $d\vec{S}$ are always out of the volume). In other words, the sum of all currents flowing into a knot is 0:

$$i - i_1 - i_2 = 0.$$

Thus let's solve the problem: Now we have 3 equations for the 3 unknowns i_1, i_1 and i_2:

$$i - i_1 - i_2 = 0 \Rightarrow i = i_1 + i_2$$
$$V - i_1 R_1 = 0 \quad \Rightarrow i_1 = \frac{V}{R_1}$$
$$i_1 R_1 - i_2 R_2 = 0 \Rightarrow i_2 = \frac{V}{R_1}\frac{R_1}{R_2} = \frac{V}{R_2}$$

$$\Rightarrow \quad i = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R}$$

But that yields also the total resistance of the circuit

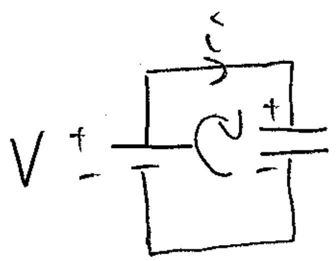
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R = \frac{R_1 R_2}{R_1 + R_2} \quad \text{(parallel resistors)}$$

NB: It's the opposite rules than for capacities!

(C) Circuits with capacitors



In the steady-state situation, there can be no current, i.e.,

$$i = 0,$$

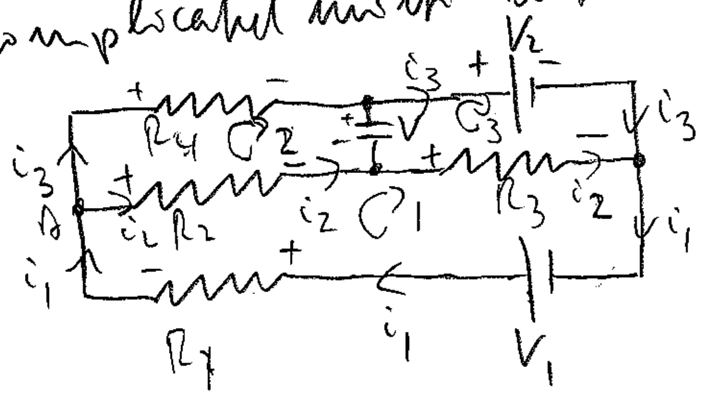
because otherwise we would pile up an infinite amount of charges on the plates. Of course our circuital law still holds here:

$$V - V_C = 0 \Rightarrow V_C = V$$

as it must be.

8.5. Elements of DC-network analysis

We like to show that with our principles we can analyse complicated multi-loop circuits for DC. Take the example



To analyze the situation assign currents through the resistors. The direction of the surface vectors, defining them by

$$i = \int_A d\vec{S} \cdot \vec{j}(\vec{r}),$$

where A is the cross-sectional area of the wire, is arbitrary. Our equations will automatically give the correct signs of the currents. Put + − signs on the resistors, indicating the direction of the so defined currents (running from + to −). Now two laws as in our simple circuits apply

(i) Integrating the $\vec{E}$ field along each circuit, where running from − to + of a battery gives the voltage of the battery, +V. Running from − to + of a resistor gives the voltage drop iR, where i is the current through the resistor with resistance R, gives 0, i.e.

$$\oint d\vec{r}\, \vec{E}(\vec{r}) = 0$$

for all circuits. (Circuital Law, also known as Kirchhoff's 2nd Law)

(ii) At each knot of the network the sum of currents vanishes (charge conservation):

$$\oint_{\partial V} d\vec{S} \cdot \vec{j} = 0 \quad \text{(continuity equation)}$$

also known as Kirchhoff's 1st law.

In our example we have for the 3 circuits:

$$1:\; V_1 - i_1 R_1 - R_2 i_2 - i_2 R_3 = 0$$
$$2:\; -V + R_2 i_2 - R_4 i_3 = 0$$
$$3:\; V - V_2 + R_3 i_2 = 0$$

Set of linear equations for unknowns V, i_1, i_2, i_3

Kirchhoff I on knot A

$$A:\; -i_1 + i_2 + i_3 = 0$$

It's most easily solved by Gauß's elimination algorithm.

Thus we sort the equations before first:

$$\begin{array}{rcl}
R_1 i_1 + (R_2 + R_3)\, i_2 & = & V_1 \quad (1)\\
R_2 i_2 - R_4 i_3 - V & = & 0 \quad (2)\\
R_3 i_2 + V & = & V_2 \quad (3)\\
-i_1 + i_2 + i_3 & = & 0 \quad (4)
\end{array}$$

To solve it we first reduce the number of equations by adding (2) to (3) $\Rightarrow$

$$\begin{array}{rcl}
(R_2 + R_3)\, i_2 - R_4 i_3 & = & V_2 \quad (2.1)\\
R_1 i_1 + (R_2 + R_3)\, i_2 & = & V_1 \quad (1)\\
-i_1 + i_2 + i_3 & = & 0 \quad (4)
\end{array}$$

From 2.1 we get

$$i_3 = \frac{(R_2 + R_3)\, i_2 - V_2}{R_4} \quad (2.2)$$

Plugging this in (4) we get:

$$-i_1 + \frac{R_2 + R_3 + R_4}{R_4}\, i_2 - \frac{V_2}{R_4} = 0 \quad (4.1)$$

multiplying (4.1) with R_1 and adding to (1) yields

$$\underbrace{\left[\frac{R_1(R_2+R_3+R_4)}{R_4}+R_2+R_3\right]}\, i_2-\frac{R_1}{R_4}V_2=V_1$$

$$\frac{R_1(R_2+R_3)+(R_1+R_2+R_3)R_4}{R_4}\, i_2=V_1+\frac{R_1}{R_4}V_2$$

$$\Rightarrow \boxed{i_2=\frac{V_1R_4+R_1V_2}{R_1(R_2+R_3)+R_4(R_1+R_2+R_3)}}$$

Plugging this into (2.2) gives

$$i_3=\frac{1}{R_4}\left[\frac{(V_1R_4+R_1V_2)(R_2+R_3)}{\underbrace{R_1(R_2+R_3)+R_4(R_1+R_2+R_3)}_{D}}-V_2\right]$$

$$=\frac{1}{R_4}\left[\frac{(V_1R_4+R_1V_2)(R_2+R_3)-V_2D}{D}\right]$$

$$=\frac{1}{\cancel{R_4}}\left[\frac{V_1\cancel{R_4}(R_2+R_3)-\cancel{R_4}(R_1+R_2+R_3)V_2}{D}\right]$$

$$\boxed{i_3=\frac{V_1(R_2+R_3)-(R_1+R_2+R_3)V_2}{R_1(R_2+R_3)+R_4(R_1+R_2+R_3)}}$$

From (4)

$$i_1 = i_2 + i_3 = \frac{V_1 (R_2 + R_3 + R_4) - (R_2 + R_3) V_2}{R_4 (R_2 + R_3) + R_4 (R_1 + R_2 + R_3)}$$

From (3) we get

$$V = V_2 - R_3 i_2$$

$$= V_2 - R_3 \frac{V_1 R_4 + R_1 V_2}{R_1 (R_2 + R_3) + R_4 (R_1 + R_2 + R_3)}$$

$$\boxed{V = \frac{[R_1 R_2 + R_4 (R_1 + R_2 + R_3)] V_2 - R_3 R_4 V_1}{R_1 (R_2 + R_3) + R_4 (R_1 + R_2 + R_3)}}$$

pages 73 + 74 not needed any more