9. The magnetic field

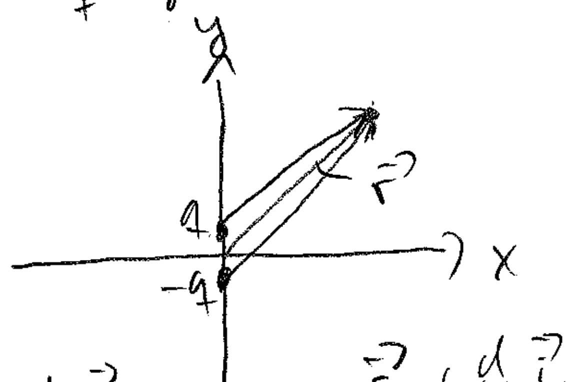
9.1 Preliminary ideas about sources of magnetic fields

So far we have looked only on electrostatic fields, i.e., electric fields which are independent of time. This was a fundamental property of matter, namely to carry an electric charge. On the most fundamental level, there are the elementary particles like electrons which are point-like and carry a (in this case negative) unit of electric charge, and the electric field by such a point charge, q, is given by Coulomb's law, which reads in our units

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3},$$

where we assume the charge to be located in the origin of the coordinate system.

Now remember the dipole which consisted of two charges, q, and $-q$ of equal magnitude but opposite sign:



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r}-\frac{d}{2}\vec{i}_y}{\left|\vec{r}-\frac{d}{2}\vec{i}_y\right|^3} - \frac{\vec{r}+\frac{d}{2}\vec{i}_y}{\left|\vec{r}+\frac{d}{2}\vec{i}_y\right|^3}\right]$$

Now suppose, we only are interested in case of $r >> d$. To simplify the field for this limiting case, we write the electric potential

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r} - \frac{d}{2}\vec{i}_y|} - \frac{1}{|\vec{r} + \frac{d}{2}\vec{i}_y|}\right)$$

Now we have

$$\left(\vec{r} - \frac{d}{2}\vec{i}_y\right)^2 = x^2 + \left(y - \frac{d}{2}\right)^2 + z^2$$

$$\approx r^2 + dy + \frac{d^2}{4}$$

$$= r^2\left(1 - \frac{dy}{r^2} + \frac{d^2}{4r^2}\right)$$

and thus, because $\frac{d}{r} << 1$:

$$\frac{1}{|\vec{r} - \frac{d}{2}\vec{i}_y|} \approx \frac{1}{r}\left[1 + \frac{1}{2}\frac{dy}{r^2} + O\left(\frac{d^2}{r^2}\right)\right]$$

In the same way

$$\frac{1}{|\vec{r} + \frac{d}{2}\vec{i}_y|} \approx \frac{1}{r}\left[1 - \frac{1}{2}\frac{dy}{r^2} + O\left(\frac{d^2}{r^2}\right)\right]$$

$$\Rightarrow \quad V(\vec{r}) \approx \frac{Q}{4\pi\varepsilon_0}\frac{dy}{r^3}$$

One defines

$$\vec{P} = Q\, d\vec{i}_y$$

as the electric dipole moment. Then the potential of a dipole reads

$$V(\vec{r}) = \frac{\vec{P}\,\vec{r}}{4\pi\varepsilon_0 r^3}$$

This expression becomes exact when we let $d \to 0$ but make $Q \to \infty$ such that $Q\,d = |\vec{P}| = \text{const.}$ One would imagine that there are pointlike particles with 0 charge but a finite electric dipole moment. So far, no such thing is known yet. However, in the sense of the approximation it's still a useful concept.

Now we can calculate the electric field

$$\vec{E} = -\operatorname{grad} V = \frac{-1}{4\pi\varepsilon_0}\left[\frac{\vec{P}}{r^3} - \frac{3}{r^4}\frac{\vec{r}}{r}(\vec{P}\vec{r})\right]$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{P}\cdot\vec{r})}{r^5}\vec{r} - \frac{\vec{P}}{r^3}\right]$$

Mini quiz

Check that for a sphere with center in the origin the total charge inside is always 0.

Solution:

We use spherical coordinates with $\vec{P}$ defining the polar axis (i.e., z-direction in our standard spherical coordinates.

(78)

Then
$$d\vec{S} = R^2 \sin\vartheta \, d\vartheta \, d\varphi, \quad \vec{i}_r \cdot \vec{P} = P\cos\vartheta$$
$$\vec{r}\cdot\vec{P} = PR\cos\vartheta \;,$$
and
$$4\pi\varepsilon_0 \vec{\Phi}^{(e)} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \, R^2 \sin\vartheta \, \vec{i}_r \left[\frac{3PR\cos\vartheta}{R^5} R \, \vec{i}_r - \frac{\vec{P}\cos\vartheta}{R^3} \right]$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \, R^2 \sin\vartheta \left[\frac{2P\cos\vartheta}{R^3} \right]$$
$$= \frac{2P}{R} 2\pi \int_0^{\pi} \sin\vartheta \cos\vartheta \, d\vartheta$$
$$= \frac{2P\pi}{R} \int_0^{\pi} d\vartheta \sin(2\vartheta) = -\frac{P\pi}{R} \cos(2\vartheta)\Big|_0^{\pi} = 0$$

We know that pieces of iron can be "magnetic", i.e., they attract other pieces of iron by the magnetic force. As you know such magnets have always a "north" and a "south" pole, and two N-poles or S-poles of two magnets repell each other and an N and an S pole attract each other.
Now you could think to divide the magnet in two pieces so that you left a single N- and S-pole. It turns out that this is impossible. You inevitably always get two new magnets with both N- and S-pole.

Thus there is nothing like a single magnetic charge. Some physicists have thought about such monopoles, but so far nobody has ever seen one.

However an electron not only carries an electric charge, but also a magnetic dipole moment, and these all line up in a iron rod to make it a permanent magnet.

We shall come back to the question of other sources of magnetic fields, most importantly the fact that electric currents always create a magnetic field, which we call $\vec{B}$.

9.2 Magnetic force on a point charge

Now we ask first a more simple question. What are the forces magnetic fields exert on particles. The most easy case is a charged particle without magnetic dipole moment. It turns out that a magnetic field only acts on moving charges, and the law is quite strange compared to the electric forces:

$$\vec{F}_m = q\,\vec{v} \times \vec{B}(\vec{r}),$$

where $\vec{r}$ is the position of the particle, q it's charge and $\vec{v} = \frac{d\vec{r}}{dt}$ it's velocity

This law is a result of many experiments and cannot be derived from more fundamental laws.

Now we can determine the unit of the magnetic field:

$$[\vec{B}] = \frac{[\vec{F}]}{[q][\vec{v}]} = \frac{N}{C\frac{m}{s}} = \frac{Ns}{Cm} = 1T \text{ (Tesla)}$$

$$= 1\,\frac{Wb}{m^2} \text{ (Weber per square m)}$$

9.3 Motion of a particle in a constant B field

Take $\vec{B}$ = const. Then the equation of motion reads

$$m\vec{a} = \vec{F}$$

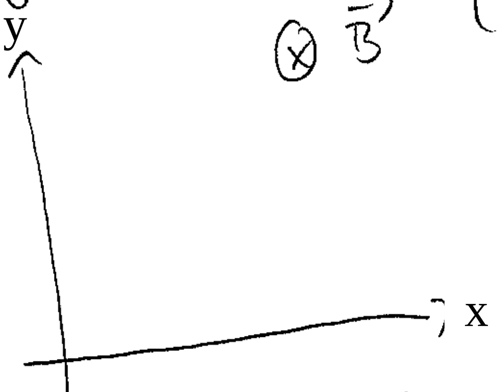
$$m\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B}$$

Let's point the $\vec{B}$-field in negative z direction. Then we have as in the book

$\otimes\ \vec{B}$ ($\vec{B}$ pointing away from you $\perp$ to the plane of the paper)

The z axis points towards you: $\odot\ \vec{i}_z$

$$\vec{B} = -B\,\vec{i}_z$$



In components our EoM reads

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\left(v_x\vec{i}_x + v_y\vec{i}_y + v_z\vec{i}_z\right)\left(-B\vec{i}_z\right)$$

$$= \frac{q}{m}\left(v_x B\,\vec{i}_y - B v_y\,\vec{i}_x\right)$$

$$\Rightarrow \frac{dv_x}{dt} = -\frac{qB}{m} v_y \quad (1)$$

$$\frac{dv_y}{dt} = +\frac{qB}{m} v_x \quad (2)$$

To solve this equation we take the time derivative of (1) and use (2) on the RHS:

$$\frac{d^2 v_x}{dt^2} = -\frac{qB}{m}\frac{dv_y}{dt} = -\left(\frac{qB}{m}\right)^2 v_x$$

The most general solution of this equation is, as should be known from PHYS-218 (harmonic oscillator):

$$v_x = -a \sin(\omega t + \varphi_0)$$

with $\omega = \frac{qB}{m}$; a and φ_0 are integration constants which we have to determine from the initial conditions.

From (1) we find

$$v_y = -\frac{1}{\omega}\frac{dv_x}{dt} = +a\cos(\omega t + \varphi_0)$$

From this we find by one more integration

$$v_x = \frac{dx}{dt} \Rightarrow x = \frac{a}{\omega}\cos(\omega t + \varphi_0) + b$$

$$y = \frac{a}{\omega}\sin(\omega t + \varphi_0) + c$$

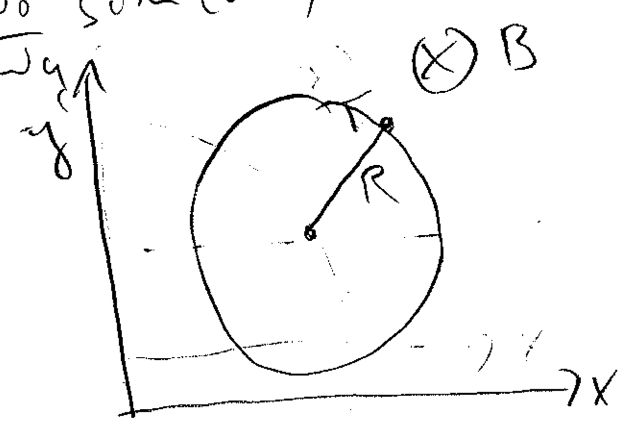
Suppose now the situation that at $t=0$, we have

$t=0$: $v_x = 0$, $v_y = v_0$

Then $\varphi_0 = 0$; $a = \frac{v_0}{\omega}$,

$$x = -\frac{v_0}{\omega}\cos(\omega t) + x_0 - \frac{v_0}{\omega}$$

$$y = \frac{v_0}{\omega}\sin(\omega t) + y_0$$



The particle goes around a circle with the "cyclotron frequency"

$\omega = \frac{qB}{m}$ which is independent of the initial conditions

which lead Lawrence to the idea of constructing the cyclotron

which finally lead to a Nobel prize for him!

We shall treat the principle of the cyclotron in the next quiz!

Note that the force along the z direction is 0:

$$m a_z = m \frac{dv_z}{dt} = 0$$

$$\Rightarrow v_z = \text{const} \Rightarrow z = v_{0z} t + z_0$$

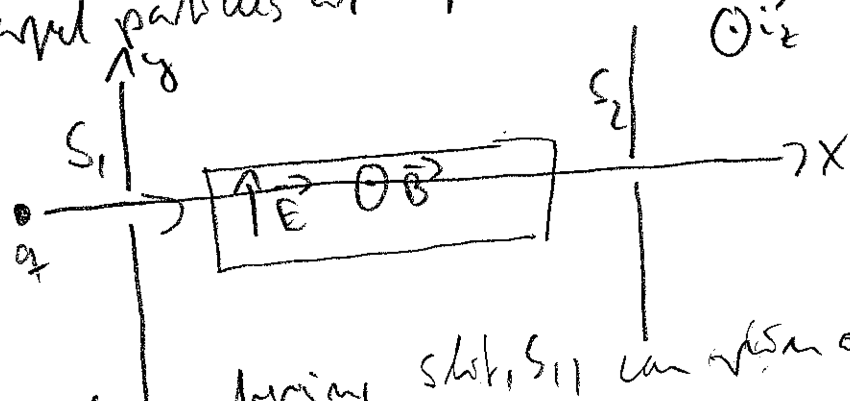
The motion of the particle is thus a spiral in space. For $v_{0z} = 0$ it's a circle in the plane $z = z_0 = \text{const} \parallel$ to the xy plane

9.4 Crossed $\vec{E}$ and $\vec{B}$ fields

Obviously, if a charge q moves through a region where both an electrical and magnetical field are present, it feels a force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

which is the complete version of the Lorentz force. As an example we look at a device, accelerator physicists use to sort out charged particles at a given speed:



A particle entering slot, S_1, can pass also through S_2, if no force deflects it. Now we have

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

In our coordinate system, we have

$$\vec{E} = E\,\vec{i}_y \; ; \; \vec{B} = B\,\vec{i}_z \; ; \; \vec{v} = v\,\vec{i}_x$$

and thus

$$\vec{F} = q(E\,\vec{i}_y + vB\,\vec{i}_x \times \vec{i}_z)$$
$$= q(E - vB)\,\vec{i}_y$$

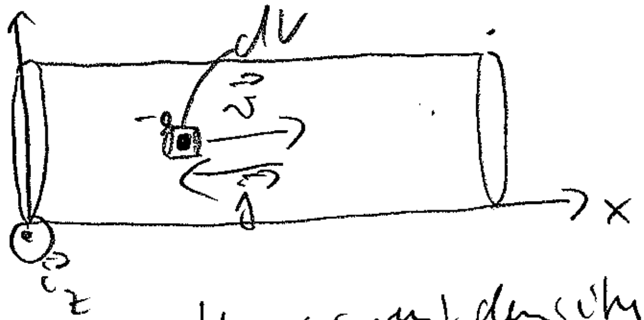
You can use the right-hand rule to verify the direction of the magnetic force.

Thus only particles, for which

$$E - vB = 0 \Rightarrow v = \frac{E}{B}$$ go through S_2

9.5 Magnetic force on a current-carrying wire

We can determine the force on a wire with a current i flowing through it by the usual trick to first determine the force element, $d\vec{F}$, on a small chunk element in the wire:



Now we have the current density vector

$$\vec{j}(\vec{r}) = -e\, n_e(\vec{r})\, \vec{v}(\vec{r}) \quad ; \quad n_e : \text{number of electrons per unit volume}$$

which was unambiguously the direction of the current. Now the magnetic force as we have learnt in the previous chapter. Now the magnetic force on the little volume element is

$$d\vec{F}(\vec{r}) = dq(\vec{r})\,\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$$
$$= -e\, n_e(\vec{r})\, dV\, \vec{v}(\vec{r}) \times \vec{B}(\vec{r})$$
$$= dV\, \vec{j}(\vec{r}) \times \vec{B}(\vec{r})$$

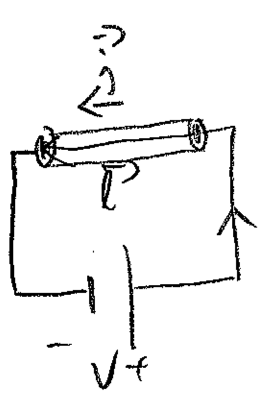
The total force is then the integral

$$\vec{F} = \int_{\text{wire}} dV\, \vec{j}(\vec{r}) \times \vec{B}(\vec{r})$$

For a thin wire we can assume that $\vec{j}$ is constant along the wire (in the steady state) and thus given by

$$\vec{j} = \frac{i}{A}\,\vec{n}, \text{ where } \vec{n} \text{ is the unit vector pointing}$$

in direction of $\vec{j}$. As we learnt before this direction is determined by the rule that the current density points along the voltage drop:



Our volume integral then reduces to a line integral, since $\vec{j}$ is always along the wire:

$$\vec{F} = \int_{\text{wire}} dV\, \vec{j} \times \vec{B}$$

$$= \int_{\text{wire}} ds\, \cancel{A}\, \frac{i}{\cancel{A}}\, \vec{n} \times \vec{B}$$

$$\vec{F} = i \int_{\text{wire}} ds\, \vec{n} \times \vec{B}$$

One sometimes writes shorter

$$ds\, \vec{n} = d\vec{s}$$

Then our law reads

$$\vec{F} = i \int_{\text{wire}} d\vec{s} \times \vec{B}(\vec{r})$$

In case of homogeneous $\vec{B}$-field and a straight wire we have

$$\vec{F} = i \int_{\text{wire}} d\vec{s} \times \vec{B} = -i\, \vec{B} \times \int d\vec{s} = -i\, \vec{B} \times \vec{l}$$

$$\vec{F} = +i\, \vec{l} \times \vec{B}$$

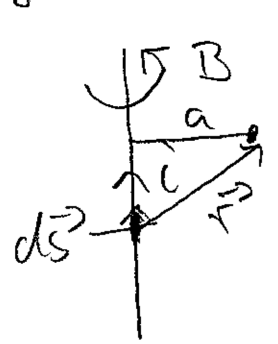
9.6 Hall effect

See problems Chapt. 9

10 Currents as sources of $\vec{B}$ and Ampère's Circuital law

10.1 Biot–Savart's law

In 1820 the Danish physicist Oersted discovered that along a current conducting wire a magnetic field is created with a magnitude proportional to the current and the inverse of the distance from the wire. The direction is given by the right-hand rule:



; $|\vec{B}| \propto \frac{i}{a}$

The direction of the current is, as usual, that of $\vec{j}$!

Almost at the same time Ampère postulated that $\vec{B}$ can be calculated by the rule that a current element $i\,d\vec{s}$ produces the field element

$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{i\,(d\vec{s}\times\vec{r})}{r^3}$$

μ_0 is determined by the definition of the unit Ampère as we shall see soon!

where $\vec{r}$ is the vector pointing from the current element to the point at which we want to calculate $\vec{B}$.

It is much easier to remember, when we do the following:

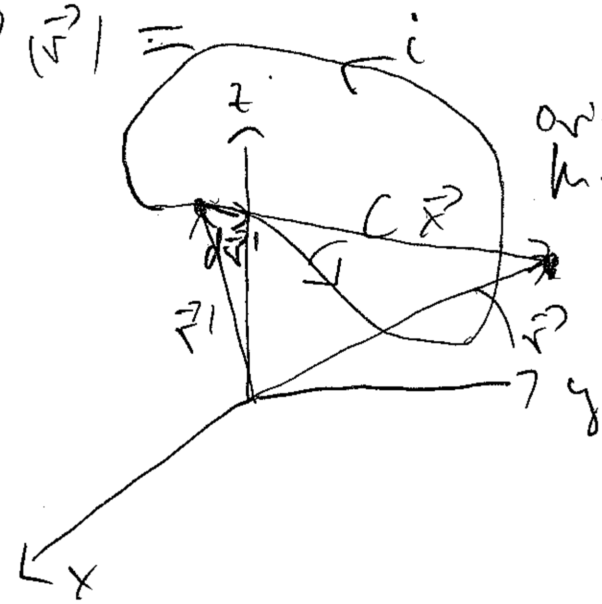
Let

$C: \vec{r}' = \vec{r}'(\lambda)$

denote the parametrization of the wire which is chosen such that $d\vec{s} = d\vec{r}' = \frac{d\vec{r}'}{d\lambda}\,d\lambda$ always points in the direction of the current in our usual sense.

Then Ampère's law reads

$\vec{B}(\vec{r}) =$

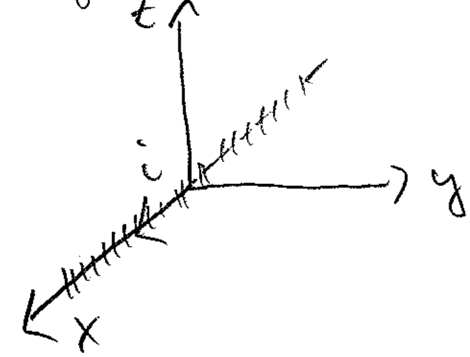


oriented such that the loop is run through for increasing λ

$$\vec{x} = \vec{r} - \vec{r}'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi} \int_C d\vec{r}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

Now let's calculate the $\vec{B}$ field of an infinitesimally long wire along the x-direction (as in the book):



Then we choose

$\vec{r}' = x' \vec{i}_x \; ; \; x' \in \mathbb{R} \Rightarrow d\vec{r}' = dx' \, \vec{i}_x$

Further $\vec{r} = x \vec{i}_x + y \vec{i}_y + z \vec{i}_z$. Then

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{\infty} dx' \, \vec{i}_x \times \frac{(x-x')\vec{i}_x + y\vec{i}_y + z\vec{i}_z}{\left[(x-x')^2 + y^2 + z^2\right]^{3/2}}$$

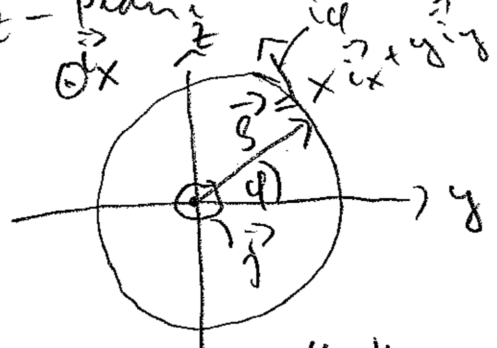
$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi}\int_{-\infty}^{\infty} dx' \, \frac{y\,\vec{i}_z - z\,\vec{i}_y}{\left[(x-x')^2+y^2+z^2\right]^{3/2}}$$

$$= \frac{\mu_0 i}{4\pi}\,\frac{x'-x}{(y^2+z^2)\left[(x'-x)^2+y^2+z^2\right]^{1/2}}\Bigg|_{x'=-\infty}^{\infty}(y\,\vec{i}_z - z\,\vec{i}_y)$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{2\pi}\,\frac{y\,\vec{i}_z - z\,\vec{i}_y}{(y^2+z^2)} \Rightarrow |\vec{B}(\vec{r})| = \frac{\mu_0 i}{2\pi}\,\frac{1}{\sqrt{y^2+z^2}}$$

as observed by Biot and Savart

As to expect from symmetry $|\vec{B}|$ is independent of z. That we also calculated the right direction becomes clear when we look on the yz-plane:



In cylinder coordinates with the x axis as cylinder axis we have

$$\vec{i}_\rho(\vec{r}) = \cos\varphi\,\vec{i}_y + \sin\varphi\,\vec{i}_z = \frac{y\,\vec{i}_y + z\,\vec{i}_z}{\rho}\;;\; \rho = \sqrt{y^2+z^2}$$

$$\vec{i}_\varphi(\vec{r}) = -\sin\varphi\,\vec{i}_y + \cos\varphi\,\vec{i}_z = \frac{-z\,\vec{i}_y + y\,\vec{i}_z}{\rho}$$

and thus, because of $\vec{r} = \rho\,\vec{i}_\rho + z\,\vec{i}_z$

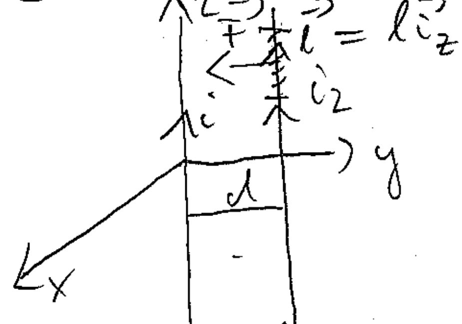
$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{2\pi\rho}\,\vec{i}_\varphi(\vec{r})$$

Thus Ampère's law gives the right field for a long straight wire according to Biot and Savart.

10.2 Force on two current-carrying wires and the definition of the Ampère

Suppose, we have the wire of the previous section and put another wire with current, i_2, in the distance d. According to the previous chapter, in each length element $\vec{l}$, there is the magnetic force

$$\vec{F} = i_2 \vec{l} \times \vec{B}$$



From the previous section we have $\vec{B}$ along the second wire:

$$\vec{F} = \frac{\mu_0 i i_2}{2\pi d} l \, \vec{i}_z \times \vec{i}_\varphi = -\frac{\mu_0 i i_2 l}{2\pi d} \vec{i}_\rho$$

"The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to $2\cdot10^{-7}$ newton per meter of length."

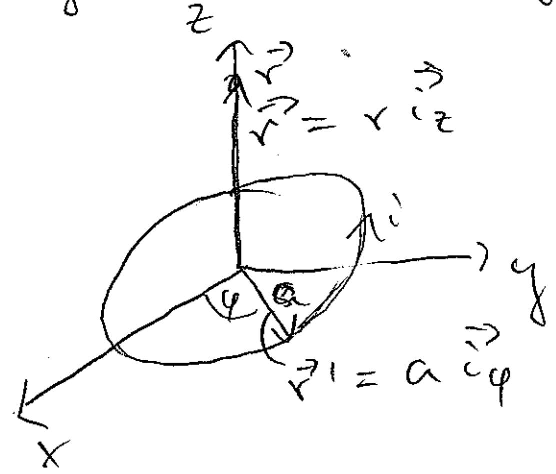
According to the formula we have just derived, this amounts to the definition of the permeability of the vacuum:

$$i = i_2 = 1A; \quad l = d = 1m$$

$$\Rightarrow 2\cdot 10^{-7} N = \frac{\mu_0 \cdot 1A^2}{2\pi} \Rightarrow \boxed{\mu_0 = 4\pi \cdot 10^{-7} \frac{N}{A^2}}$$

10.3 The Field of a Current carrying loop

Next we calculate the magnetic field of a current carrying loop, a circle of radius a in the xy plane, along the z axis:



$$\vec{r}\,' = a\,\vec{i}_\varsigma(\varphi)$$

$$d\vec{r} = a\,d\varphi\,\frac{d\vec{i}_\varphi}{d\varphi} = a\,d\varphi\,\vec{i}_\varphi(\varphi)$$

$$\vec{B} = \frac{i\mu_0}{4\pi}\int_0^{2\pi} d\varphi\, a\,\vec{i}_\varphi(\varphi) \times \frac{\vec{r}-\vec{r}\,'}{(a^2+r^2)^{3/2}}$$

$$\vec{i}_\varphi(\varphi) \times (\vec{r}-\vec{r}\,') = \vec{i}_\varphi(\varphi) \times [r\,\vec{i}_z - a\,\vec{i}_\varsigma]$$

$$= \vec{r}\,\vec{i}_\varsigma(\varphi) + a\,\vec{i}_z$$

Since $\int_0^{2\pi} d\varphi\,\vec{i}_\varsigma(\varphi) = \int_0^{2\pi} d\varphi\,[\cos\varphi\,\vec{i}_x + \sin\varphi\,\vec{i}_y] = 0$

we find

$$\vec{B}(r\vec{i}_z) = \frac{i\mu_0}{4\pi}\int_0^{2\pi} d\varphi\,\frac{a^2\vec{i}_z}{(a^2+r^2)^{3/2}} = \frac{i\mu_0}{2}\,\frac{a^2}{(a^2+r^2)^{3/2}}\,\vec{i}_z$$

For $r \gg a$, we obtain

$$\vec{B}(r\vec{i}_z) = \frac{\mu_0}{2\pi}\,\frac{\vec{M}}{r^3}$$

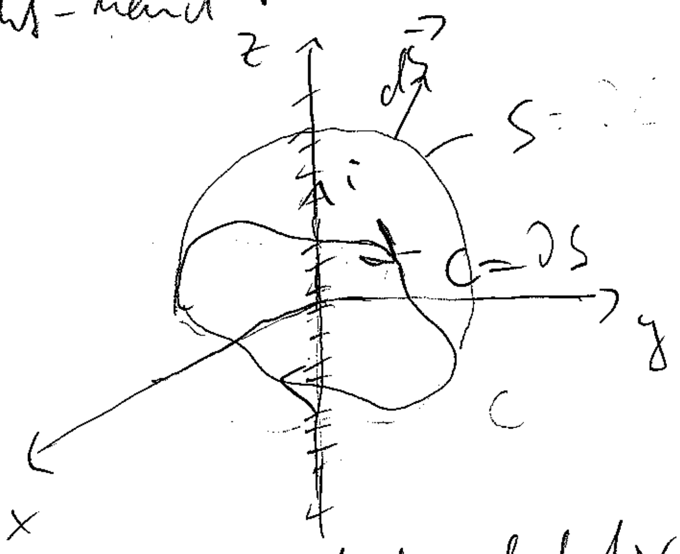
where the magnetic moment of the loop is defined by
$$\vec{\mu} = A\, i$$

10.4 Ampère's Circuital Law

Now we look at an analogous law for $\vec{B}$ as we found in terms of Gauss's law for $\vec{E}$. Here, we shall not prove it in full generality, but stick to the case of an infinitely long wire.

The statement is:

For any closed curve C, encircling the wire, we have
$$\oint_C d\vec{r}\, \vec{B}(\vec{r}) = \mu_0\, i_{\text{incl}} = \mu_0 \int_S d\vec{S}\, \vec{j}(\vec{r}),$$
where i is the total current running through any surface with C as boundary. The surface elements must be oriented relative to the orientation of the curve C corresponding to the right-hand rule:

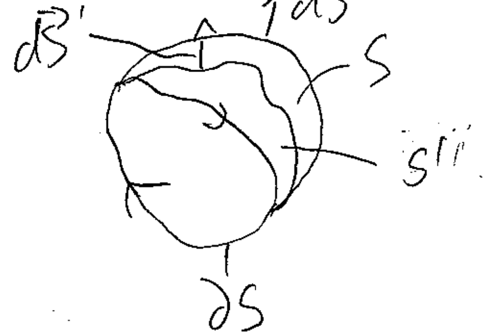


The statement is in fact valid for all stationary current density vectors, $\vec{j}(\vec{r})$. Ampère's law describes all $\vec{B}$ fields created by such currents:
$$\oint_{\partial S} d\vec{r}\, \vec{B}(\vec{r}) = \mu_0 \int_S d\vec{S}\, \vec{j}(\vec{r})$$
($\vec{B}$ field is <u>not</u> a conservative field!)

It is important to note that this is only a consistent statement if charge is conserved. Suppose we choose another surface S' with the same boundary curve: $\partial S' = \partial S$. Then we must have

$$\oint_{\partial S} d\vec{r}\,\vec{B}(\vec{r}) = \mu_0 \int_S d\vec{S}\,\vec{j}(\vec{r}) \qquad (A)$$

$$\stackrel{\boxed{\text{because } \partial S' = \partial S}}{=} \oint_{\partial S'} d\vec{r}\,\vec{B}(\vec{r}) = \mu_0 \int_{S'} d\vec{S}\,\vec{j}(\vec{r}) \qquad (B)$$



Now $S - S'$ is a closed surface, "$-S'$" meaning the same surface as S' but with the opposite orientation. From Ampère's law we infer from (A) and (B) that

$$\oint_{S-S'} d\vec{S}\,\vec{j}(\vec{r}) = 0$$

Since on the other hand we can draw any closed surface and draw an arbitrary closed curve on it, we must have

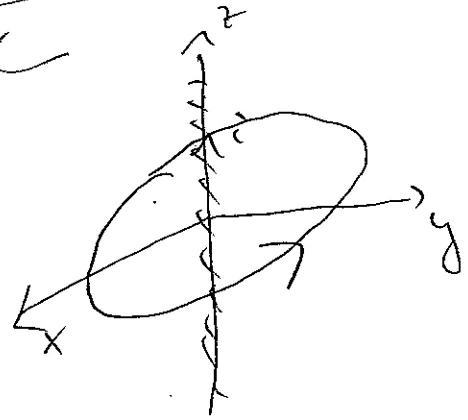
$$\oint_S d\vec{S}\,\vec{j}(\vec{r}) = 0$$

for any closed surface. As we know from the previous chapters, that's precisely stating the conservation of electric charge.

for steady-state currents

10.5 Proof of Ampère's circuital law for the $\vec B$ field of an infinite wire

(a) Circle $\perp$ to the wire with the wire running through its center



$$\vec B = \frac{\mu_0}{2\pi}\frac{i}{\rho}\vec i_\varphi(\varphi)$$

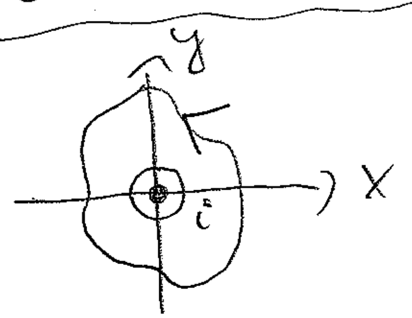
$$C: \vec r(\varphi) = a\,\vec i_\rho \Rightarrow \frac{d\vec r}{d\varphi} = a\,\vec i_\varphi$$

$$\vec B[\vec r(\varphi)] = \frac{\mu_0}{2\pi}\frac{i}{a}\vec i_\varphi(\varphi)$$

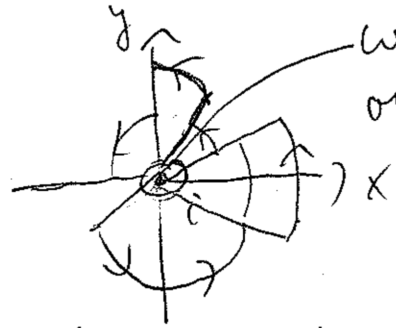
$$\Rightarrow \oint_C d\vec r\,\vec B(\vec r) = \int_0^{2\pi} d\varphi\, a\frac{\mu_0}{2\pi}\frac{i}{a} = \mu_0 i \quad ☺$$

(because $\vec i_\varphi \cdot \vec i_\varphi = 1$).

(b) Arbitrary curve in the plane $\perp$ to wire containing the wire



We can approximate this path arbitrarily precise with piecewise drawn sections of a circle with the wire as center, connected by straight radial lines:



Since all these circle segments give always a contribution

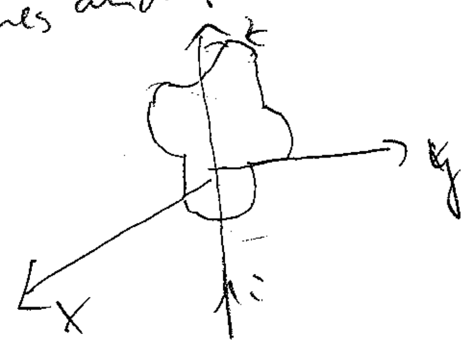
$$\frac{\mu_0}{2\pi} i \, \Delta\varphi.$$

and the radial lines give 0, since for them $d\vec{r} \sim \vec{i}_g$ and $\vec{B} \sim \vec{i}_\varphi$ and $\vec{i}_\varphi \cdot \vec{i}_g = 0$, we again find after summing over all segments:

$$\oint d\vec{r} \cdot \vec{B}(\vec{r}) = \mu_0 i$$ ☺

(c) For a non plane curve containing the wire in its interior

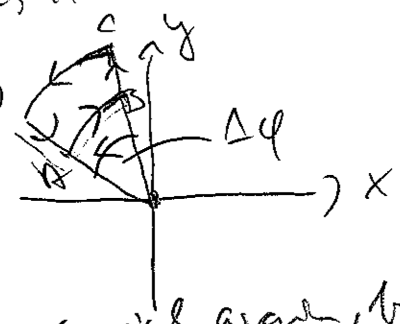
Then we can use a similar argument. We approximate the path by circles in planes ⊥ to the wire connected by radial lines in this planes and lines ∥ to the wire:



The pieces $\parallel$ in $\vec{i}_\rho$ and $\vec{i}_z$ directions give 0 in the integral, because $d\vec{r} \perp \vec{B} \sim \vec{i}_\varphi$. The circular segments sum up to $i\mu_0$ as in case (b).

(d) Curves not containing the wire

This can be approximated by segments and straight radial and lines ∥ to the wire as well:



The radial pieces vanish again, because $d\vec{r}\,\vec{B} = 0$ along those. For the circle segments we have

$$\int_C^D d\vec{r} \cdot \vec{B}(\vec{r}) = \frac{\mu_0 i}{2\pi} \Delta\varphi$$

but
$$\int_A^B d\vec{r} \cdot \vec{B}(\vec{r}) = -\frac{\mu_0 i}{2\pi} \Delta\varphi$$

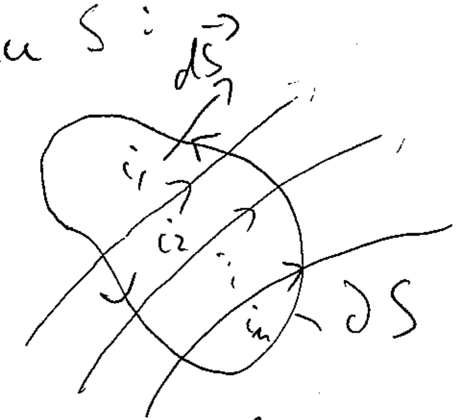
and thus
$$\oint_C d\vec{r}\,\vec{B}(\vec{r}) = 0$$

This is a rather complete proof for Ampère's law for the $\vec{B}$-field of an infinite current carrying wire, but note that Ampère's circuital law is the basic law for the interplay between currents and $\vec{B}$ fields. Because if is true for all surfaces S with arbitrary boundaries ∂S,

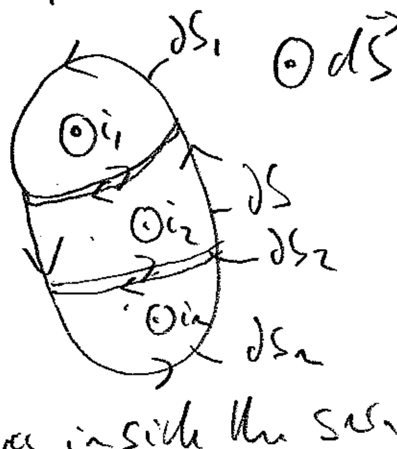
it can be made a local law, connecting $\vec{B}$ and $\vec{j}$.
We won't go on to this in this lecture, but rather use it
for some symmetric but rather important situations.

10.6 Superposition

If we have more than one current-conducting wire, the
total $\vec{B}$ field is given by the sum of the $\vec{B}$ fields. This
can be seen with Ampère's circuital law as follows:
Consider various currents $i_1, i_2, \ldots, i_n$ running through
a surface S:



Drawing this in a plane makes it easier to see, but it is
valid for any surface, not only plane ones!



Since the paths inside the surface are run through twice
in opposite direction, the contributions along them cancel
out. Thus we can write

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$$\int_{\partial S} d\vec{r}\,\vec{B}_{tot} = \int_{\partial S_1} d\vec{r}\,\vec{B}_{tot} + \int_{\partial S_2} d\vec{r}\,\vec{B}_{tot} + \dots + \int_{\partial S_n} d\vec{r}\,\vec{B}_{tot}$$

$$= \sum_{k=1}^{n} \int_{\partial S_k} d\vec{r}\,\vec{B}_{tot}(\vec{r}) = r_0 \sum_{k=1}^{n} i_k$$

On the other hand, if we denote the $\vec{B}$-field made up by the current i_k with $\vec{B}_k$, we have

$$\int_{\partial S_k} d\vec{r}\,\vec{B}_{k'} = \begin{cases} r_0\, i_k & \text{for } k = k' \\ 0 & \text{for } k \neq k' \end{cases}$$

But this also holds true for the boundary ∂S:

$$\int_{\partial S} d\vec{r}\,\vec{B}_{k} = r_0\, i_k$$

and thus

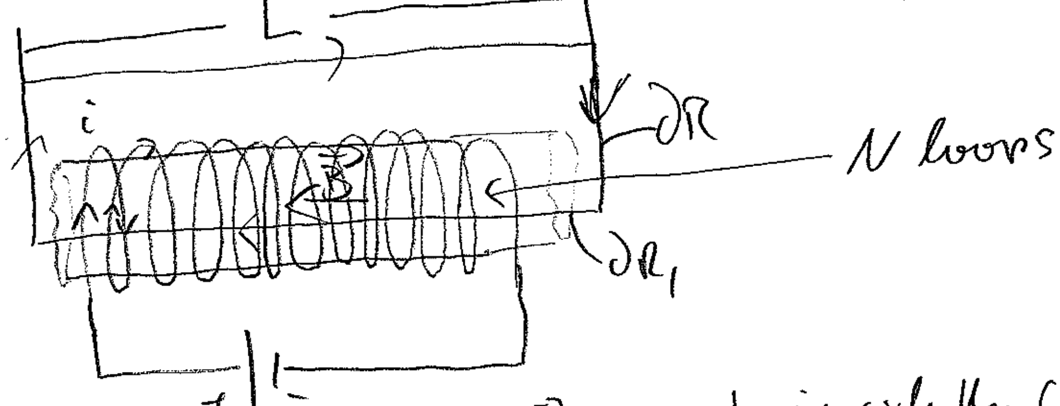
$$\int_{\partial S} d\vec{r} \sum_{k=1}^{n} \vec{B}_k = r_0 \sum_{k=1}^{n} i_k$$

So we find, since this is valid for any closed curve ∂S

$$\vec{B}_{tot} = \sum_{k=1}^{n} \vec{B}_k$$

10.7 Applications of Ampère's Law

(a) The long solenoidal coil



Assumption: $\vec{B} = B_x \vec{i}_x$ = const. inside the cylinder, while $B = 0$ outside. Then we choose the rectangle with boundary ∂R oriented in the correct way relative to the currents running through it. If the coil winds N turns around, then we have according to Ampère's Law

$$\oint_{\partial R} d\vec{r}\, \vec{B}(\vec{r}) = N \mu_0 i.$$

Only the side inside the cylinder contributes, but

∂R_1: $\vec{r} = -\lambda \vec{i}_x$ with $\lambda \in (0, L) \Rightarrow d\vec{r} = -d\lambda\, \vec{i}_x$

and thus

$$\oint_{\partial R} d\vec{r} \cdot \vec{B}(\vec{r}) = - \int_0^L d\lambda\, B_x = -LB$$

and thus

$$\boxed{B_x = -\frac{N \mu_0 i}{L}}$$

This means the field points to the left as we already could guess from the direction of the current elements with help of the right-hand rule. The magnitude is

$$\boxed{|\vec{B}| = \frac{N \mu_0 i}{L}}$$

(b) Field of an infinitely long wire of finite cross section

Since

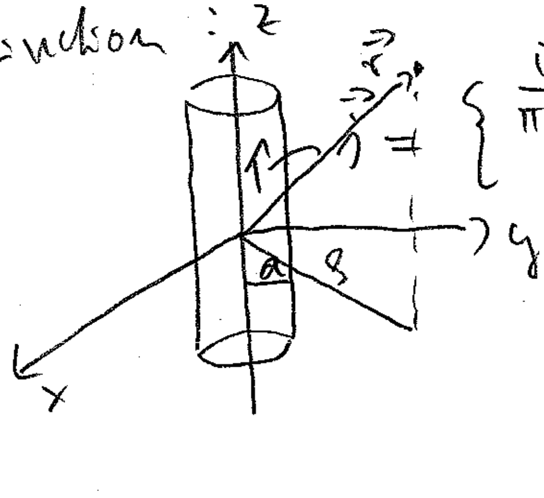
$$\oint_S d\vec{S}\,\vec{B} = 0$$

for any closed surface S, the field lines must be closed. From the cylindric symmetry they must be circles around the wire. Thus we make the ansatz

$$\vec{B} = B_\varphi(\rho)\,\vec{i}_\varphi \quad (e)$$

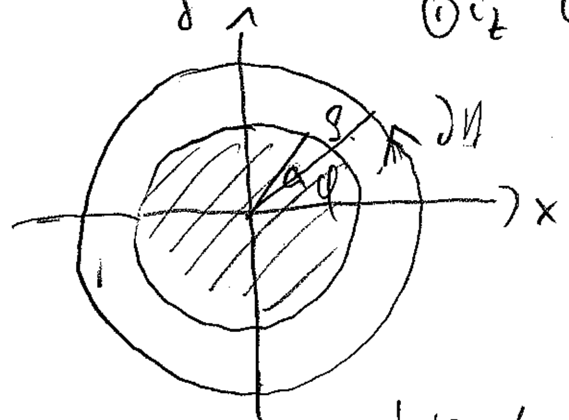
using the usual cylinder coordinates. We assume that the current density is uniform and pointing in positive z direction:

$$\vec{j}(\vec{r}) = \begin{cases} \frac{i}{\pi a^2}\,\vec{i}_z & \text{for } \rho = \sqrt{x^2+y^2} < a \\ 0 & \text{for } \rho > a \end{cases}$$



Now we use Ampère's law with circles of radius ϱ around the cylinder:

$\odot \vec{i}_z \quad \odot \vec{j} \quad \odot d\vec{A}$



Our line ∂A is parametrized as

$$\partial A: \vec{r} = \vec{r}(\varphi) = \varrho\, \vec{i}_\varrho(\varphi) = \varrho\,[\cos\varphi\, \vec{i}_x + \sin\varphi\, \vec{i}_y]$$

$$d\vec{r} = d\varphi\, \varrho\, \vec{i}_\varphi(\varphi) = d\varphi\, \varrho\,(-\sin\varphi\, \vec{i}_x + \cos\varphi\, \vec{i}_y)$$

For the surface integral we have

$$d\vec{A} = \varrho'\, d\varrho'\, d\varphi\, \vec{i}_z \quad \text{with } \varrho' \in (0, \varrho)$$

Ampère's law reads as

$$\int_{\partial A} d\vec{r}\, \vec{B}(\vec{r}) = \mu_0 \int_A d\vec{A} \cdot \vec{j}(\vec{r})$$

So we find

$$\int_{\partial A} d\vec{r}\, \vec{B}(\vec{r}) = \int_0^{2\pi} \underbrace{d\varphi\, \varrho\, \vec{i}_\varphi(\varphi)}_{d\vec{r}} \cdot \underbrace{B_\varphi(\varrho)\, \vec{i}_\varphi(\varphi)}_{\vec{B}(\vec{r})}$$

$$= \varrho\, B_\varphi(\varrho) \cdot \int_0^{2\pi} d\varphi = 2\pi \varrho\, B_\varphi(\varrho)$$

and

$$\int_A d\vec{A} \cdot \vec{j}(\vec{r}) = \int_0^{\varrho} d\varrho' \int_0^{2\pi} d\varphi\, \varrho'\, \vec{i}_z \cdot \vec{j}(\vec{r})$$

Now we have to distinguish two cases:

(α) $\varrho < a \Rightarrow \vec{j} = \frac{i}{\pi a^2}\vec{i}_z$ for all $\varrho' \Rightarrow$

$$\int_A d\vec{A} \cdot \vec{j} = \int_0^{\varrho} d\varrho' \int_0^{2\pi} d\varphi\, \varrho' \frac{i}{\pi a^2} = \frac{i}{\pi a^2} \int_0^{\varrho} \varrho' d\varrho' \cdot 2\pi$$

$$= \frac{2i}{a^2} \frac{1}{2} \varrho^2 = \frac{i \varrho^2}{a^2}$$

$$\Rightarrow 2\pi \varrho\, B_\varphi(\varrho) = \frac{\mu_0 i \varrho^2}{a^2}$$

$$\Rightarrow \boxed{B_\varphi(\varrho) = \frac{\mu_0 i \varrho}{2\pi a^2} \;;\; \varrho < a}$$

(β) $\varrho > a \Rightarrow \vec{j} = \frac{i}{\pi a^2}\vec{i}_z$ for $\varrho' \in (0, a)$

$$\Rightarrow \int_A d\vec{A} \cdot \vec{j} = \frac{2i}{a^2} \int_0^a d\varrho' \varrho' = i$$

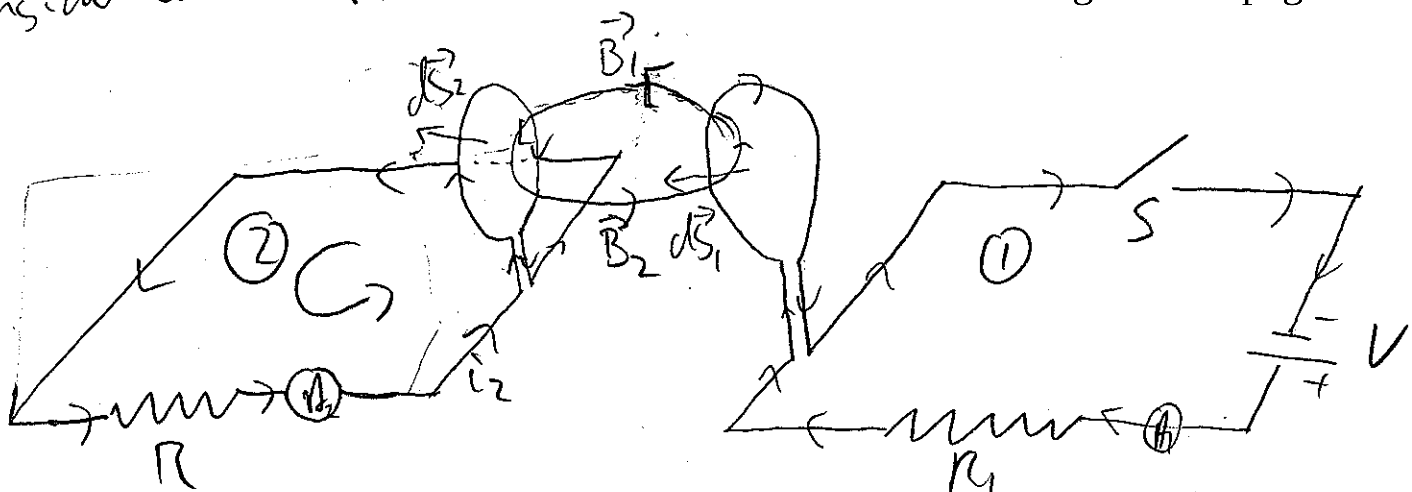
$$\Rightarrow \boxed{B_\varphi(\varrho) = \frac{\mu_0 i}{2\pi \varrho} \;;\; \varrho > a}$$

11. Faraday's Law of induction

In this chapter we begin the investigation of time-dependent electromagnetic phenomena. We need some experimental input first, as usual in physics!

11.1 The experiment by Faraday and Henry

Consider two loops as follows see a better drawing on next page



We assume that the whole setup is given in this way for some time and we assume, no currents are flowing in the circuits. Now we close the switch, S, in circuit 1. Then a current begins to flow. Clearly that's a time dependent current, given by some current density vector

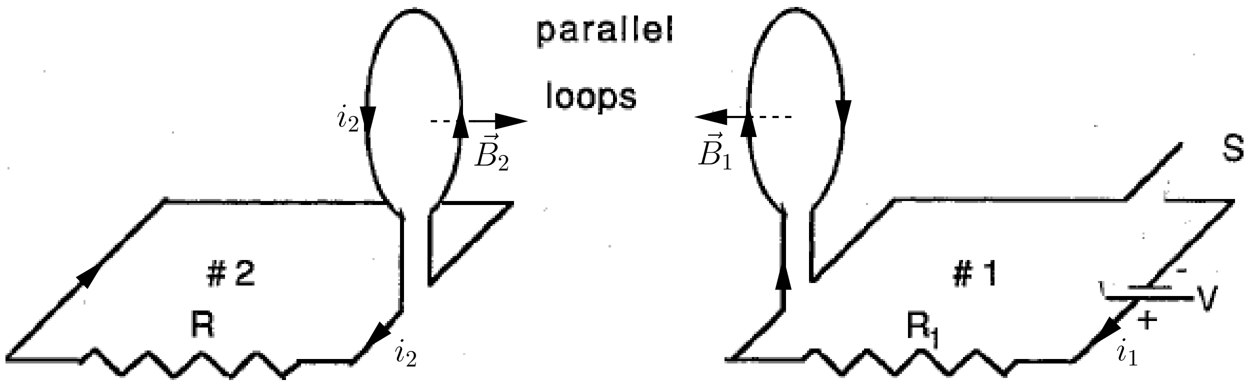
$$\vec{j}_1(t, \vec{r}) \quad \text{(current in circuit 1)}$$

only after a long time, the total current will be

$$i_1(t \to \infty) = \frac{V}{R_1}$$

The current won't jump instantaneously to this value. Now the remarkable finding by the physicists Faraday and Henry has been that there is a current "induced"

The Faraday experiment



- Faradays Law of induction

$$\oint_{\partial\#2} \mathrm{d}\vec{r} \cdot \vec{E}(t,\vec{r}) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\#2} \mathrm{d}\vec{S} \cdot \vec{B}(t,\vec{r}) \tag{1}$$

- When closing the switch, in circuit #1 a **time-dependent** current starts to run
- this induces a **time-dependent** magnetic field $\vec{B}_1$, reaching through the loop in circuit #2
- there it induces an EMF, resulting in a current which induces a magnetic field $\vec{B}_2$ counteracting the built-up of $\vec{B}_1$ in the loop of circuit #2.

in circuit 2, although there is no battery included there. Thus there must be an electromotive force (EMF) induced in circuit 2 along the resistor R leading to a current

$$i_2(t) = \frac{\mathcal{E}}{R} \quad (\mathcal{E}: \text{EMF along } R)$$

measured with the amp meter A_2.

This current only rises during the current i_1 is changing on circuit 1. Faraday thus concluded that the EMF must be due to the time changing $\vec{B}_1$ field produced by the time varying current in circuit 1.

11.2 Faraday's Law of induction

Faraday found out, performing his experiment, that the electromotive force is given by the law

$$\oint_{\partial S} d\vec{r}\, \vec{E}(t,\vec{r}) = -\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}(t,\vec{r})$$

Important: ∂S and S are oriented relative to each other by the RHR*

We look at the loop in circuit 1. Then we have a field $\vec{B}_1$ which is directed as indicated. In circuit ② this induces the following EMF:

$$\int_{\partial S_2} d\vec{r}\, \vec{E}(t,\vec{r}) = +M\frac{di_1}{dt}$$

with a positive M. But since $\frac{di_1}{dt}$ is positive at $t=0$, we have the current i_2, given by

$$\int_{\partial S_2} d\vec{r}\, \vec{E}(t,\vec{r}) = R\, i_2 = M\frac{di_1}{dt}$$

*Remember: RHR = Right-Hand Rule

$d\vec{S}$ ("direction of thumb out of paper")

∂S ("direction of fingers counter-clock-wise")

Thus U_2 is positive and running thus in the direction as indicated. This gives a $\vec{B}_2$ in loop 1 giving a induced current against the direction of the current by the battery in curve (2). That's known as

Lenz's Law

If a current is induced by some change, its direction is such that it opposes the change.

Note: It would be a disaster, if the sign in Faradays Law would be +. Then we would produce an infinite current in both loops by just using a tiny amount of energy from the battery to begin with the process. This would violate the law of energy conservation, since we would produce an infinite amount of heat in the resistors out of nothing. This would be in contradiction to experience!

It is important to note that due to

$$\oint_{\partial S} d\vec{r} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{S} \cdot \vec{B} := -\frac{d}{dt} \Phi_{\vec{B}}$$

$\vec{E}$ is in general not a conservative field. Whenever the magnetic flux is changing with time. Of course our laws for steady-state currents or static charges is still correct, since then $\Phi_{\vec{B}}$ is independent of time. Note also that $\Phi_{\vec{B}}$ can change due to different reasons

(1) $\vec{B}$ changes with time

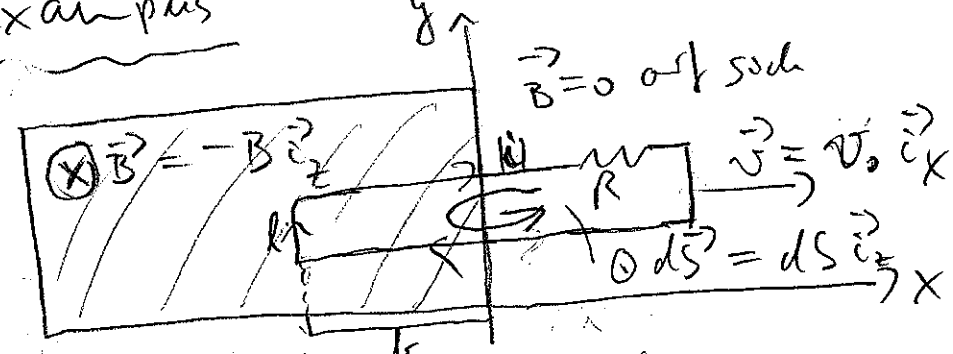
(2) The surface area S changes with time

(3) The direction of the surface normal relative to $\vec{B}$ changes

(4) any combination of this

11.3 Examples

(a)



$$\Phi_{\vec{B}} = \int_S d\vec{S}\cdot\vec{B} = -Blb$$

$$\frac{d\Phi_{\vec{B}}}{dt} = -Bl\frac{db}{dt} = +Blv_0$$

$$\oint_{\gamma_S} d\vec{r}\,\vec{E} = -iR = -\frac{d\Phi_{\vec{B}}}{dt} = -Blv_0$$

$$\Rightarrow \quad i = +\frac{Blv_0}{R} > 0 \Rightarrow \text{current runs clockwise!}$$

The force due to this current on the loop is thus

$$\vec{F} = l|i|\vec{i}_y\times\vec{B} = -l|i|B\,\vec{i}_x$$

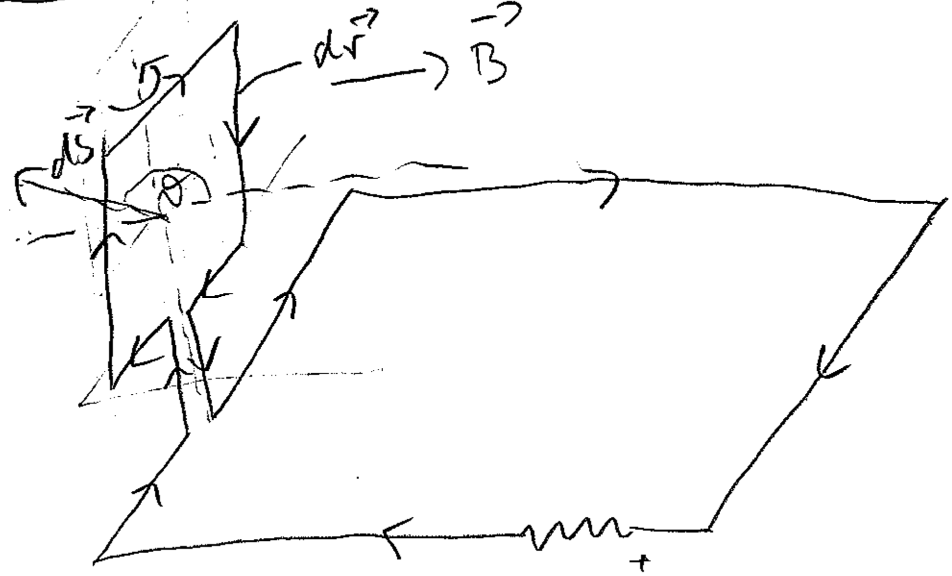
Thus to keep $v_0 =$ const. one has to pull with this force. The power used is

$$-\vec{F}\vec{v}_0 = l|i|Bv_0 = R|i|^2$$

as it must be, since this is precisely the heat power produced on the resistor.

The example shows again Lenz's law: The induced current is directed such to hinder the change in magnetic flux caused by decreasing the area the $\vec{B}$ field intersects.

(15) A simple generator



$$\Phi_B = \int d\vec{S}\cdot\vec{B} = AB\cos(\omega t)$$

$$\oint d\vec{r}\,\vec{E} = iR = -\frac{d\Phi}{dt} = AB\omega\sin(\omega t)$$

$$\Rightarrow \boxed{i(t) = \frac{AB\omega}{R}\sin(\omega t)}$$

The force on the loop is such that it works against the rotation which is in accordance with Lenz's Law!

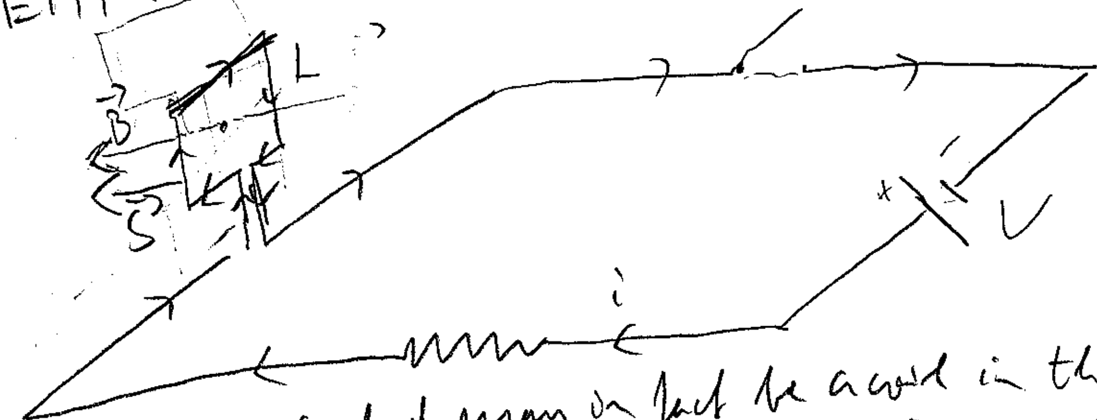
11.4 Mutual inductance and self-inductance

Coming back to our discussion of Faraday's experiment, we note that the current change in loop ① induced a current in loop ② which was due to the change in the magnetic flux $\Phi_{\vec{B}}^{(2)}$. In general it is difficult to calculate the flux for the given geometry, but we know that $\vec{B}\propto i_1$ and thus also $\Phi_{\vec{B}}$ is $\propto i_1$:

$$\Phi^{(2)}_{\vec{B}} = M i_1$$

The sign has to be determined in each case by using the conventions for the orientation of the boundary line and the normal of the surface. In any case one defines $I > 0$. Often one can also directly apply Lenz's rule!

An example is self inductance. It gives rise to induction of an EMF in a coil itself as follows:



In the loop L (which may in fact be a coil in the circuit. Now we calculate the magnetic flux, if there is a current running as indicated (at time t). The magnetic flux through the coil itself is again $\propto i$:

$$\Phi_{\vec{B}} = L i$$

L is called self-inductance of the coil. The units are

$$[\Phi_{\vec{B}}] = [B][A] = T\,m^2 = \frac{Wb}{m^2}\,m^2 = Wb$$

$$\Rightarrow [L] = \frac{Wb}{A} = H \quad (\text{Henry})$$

As an example we calculate the time dependence of the current. We use our orientation rule of ∂S and S in Faraday's law to set the sign right:

$$\Phi_{\vec{B}} = +L i > 0$$

$$\Rightarrow -L\frac{di}{dt} = \oint_{\partial S} d\vec{r}\cdot\vec{E} = -V + iR$$

$$\frac{di}{dt} = -\frac{R}{L} i + \frac{1}{L} V$$

To solve the equation, we note that if i_1 and i_2 are two solutions we have

$$\left.\begin{aligned}\frac{di_1}{dt} &= -\frac{R}{L} i_1 + \frac{1}{L} V \\ \frac{di_2}{dt} &= -\frac{R}{L} i_2 + \frac{1}{L} V\end{aligned}\right\} \Rightarrow \frac{d(i_1 - i_2)}{dt} = -\frac{R}{L}(i_1 + i_2)$$

This means we need a general solution i_1 for $V=0$ and a special solution i_2 for $V \neq 0$. It's clear that the latter solution is given by the steady state

$$i_2 = \text{const} \Rightarrow 0 = -\frac{R i_2}{L} + \frac{1}{L} V \Rightarrow \boxed{i_2 = \frac{V}{R}}$$

$$\frac{di_1}{dt} = -\frac{R}{L} i_1 \Rightarrow \frac{1}{i_1}\frac{di_1}{dt} = -\frac{R}{L}$$

$$\Rightarrow \frac{d}{dt} \ln(i_1) = -\frac{R}{L}$$

$$\Rightarrow \ln\left(\frac{i_1}{a_0}\right) = -\frac{R}{L} t$$

$$\Rightarrow i_1 = a \exp\left(-\frac{R}{L} t\right) \quad (a = \text{const})$$

Thus the solution is

$$i = a \exp\left(-\frac{R}{L} t\right) + \frac{V}{R}$$

If the switch is closed at $t=0$, we have $i(0) = 0 \Rightarrow a = -\frac{V}{R}$

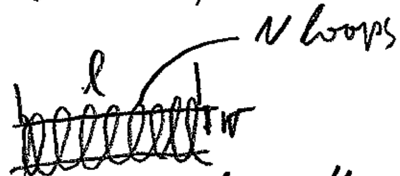
$$\boxed{i = \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L} t\right)\right]}$$

11.5 Self-inductance of a solenoidal coil

A solenoidal coil with a current i has the B field

$$B = \mu_0 \frac{N}{l} i$$

(remember our derivation with help of Ampère's law!).



The flux through 1 loop is

$$\Phi_{\vec{B}}^{(1\text{ loop})} = A B = \pi r^2 B$$

Thus integrating along the wire in Faraday's law involves the total flux through all the N windings!

$$\Phi_{\vec{B}} = N \Phi_{\vec{B}}^{(1\text{ loop})} = N \pi r^2 B = \mu_0 N^2 \pi \frac{r^2}{l} i$$

$$=: L i$$

$$\Rightarrow \boxed{L = \mu_0 N^2 \pi \frac{r^2}{l}}$$ (solenoidal coil)

12. AC Circuits

Now we are ready to discuss circuits with the following elements:

Resistor: Resistance R (Unit Ohm $= \Omega$) — R

Capacitor: Capacitance C (Unit Farad $= F$) — C

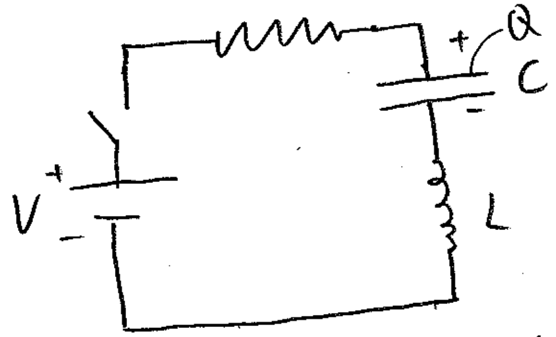
Coil: Self-inductance L (Unit Henry $= H$) — L

Now we shall study some examples, how to find the equations for time-changing currents, voltages etc.

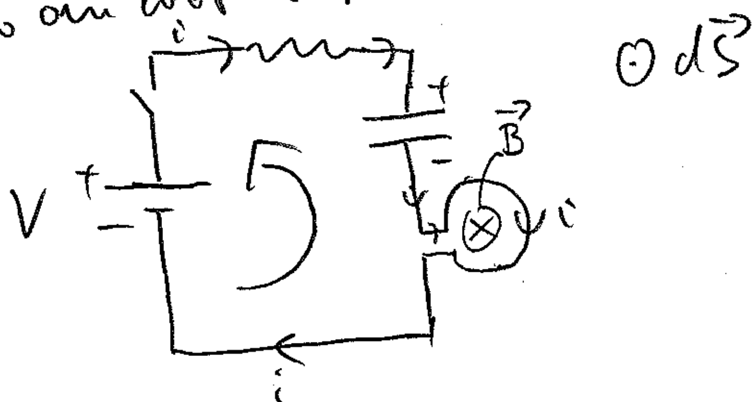
12.1 RLC circuit connected to battery

We always use Faraday's law to get the right signs!

Let's consider the simple circuit



To find the right signs, ~~we~~ we simplify the coil to one loop but associate its self-inductance with it



We assume a current in an arbitrary direction. Here it is customary to use the direction as assumed, because of the battery which is a constant voltage. Then we apply Faraday's law. Here we go counter clockwise along the current:

$$\oint_{\partial S} d\vec{r} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{S} \cdot \vec{B}$$

Since we go against the current, we have

$$\oint_{\partial S} d\vec{r} \cdot \vec{E} = V - iR - \frac{1}{C}Q$$

Since $\vec{B}$ is such the loop goes into the plane, but $d\vec{S}$ is going out, we have

$$\Phi_B = -Li$$

and thus

$$V - iR - \frac{1}{C}Q = +L\frac{di}{dt}$$

To get a differential equation, we use

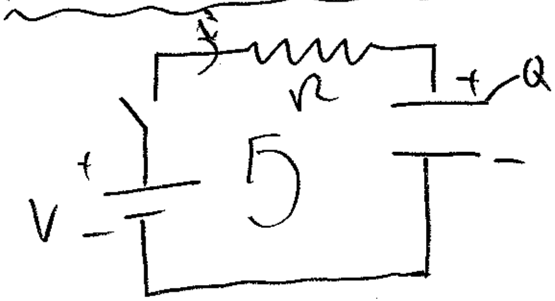
$$i = +\frac{dQ}{dt}$$

The sign is determined by the fact that Q on the + plate of the capacitor grows, if the current goes as indicated.
Thus, we obtain

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + CQ = V$$

We shall discuss the solution of this equation in the next quiz, because it is a very important case and it's covered only uncompletely in the book.

12.2 The RC Circuit



Here we neglect the self inductance, and Faraday's law gives

$$\oint_{\partial S} d\vec{r} \cdot \vec{E} = 0$$

That's true in the steady state. Then we have

$$\oint d\vec{r} \cdot \vec{E} = V - \frac{Q}{C} - iR = 0$$

Again $i = \dot{Q} := \frac{dQ}{dt}$

$$\Rightarrow \quad R\dot{Q} + \frac{Q}{C} = V$$

This equation we can solve easily. Its general solution is the sum of the general solution of the homogeneous equation, Q_H, and a particular solution of the inhomogeneous eq.

$$R\dot{Q}_H + \frac{Q_H}{C} = 0$$

We make the ansatz

$$Q_H(t) = a \exp(\lambda t) \quad \text{with } a = \text{const.}$$

$$\Rightarrow \quad R a \lambda \exp(\lambda t) + \frac{1}{C} a \exp(\lambda t) = 0$$

$$\Rightarrow \quad R\lambda + \frac{1}{C} = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{RC}$$

Thus

$$Q_H(t) = a \exp\left(-\frac{t}{RC}\right),$$

Particular solution of the inhom. eq

$$R\,\dot{Q}_I + \frac{1}{C} Q_I = V$$

Since $V = \text{const.}$ It is clear that the steady-state solution $Q_I = \text{const}$. The equation tells us of course the right value:

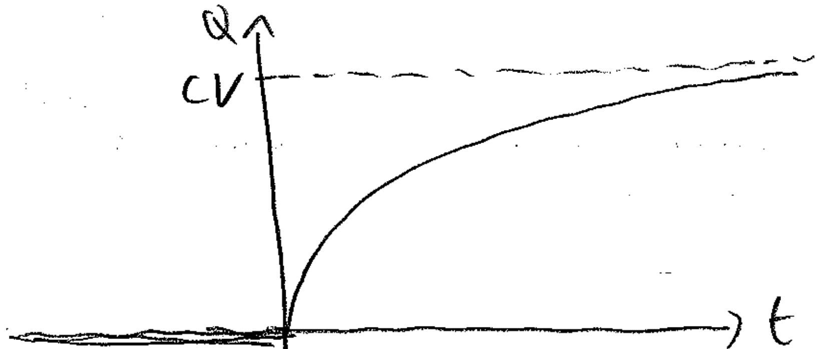
$$Q_I = C\,V$$

General solution of the eq.

$$Q(t) = Q_H(t) + Q_I(t) = a \exp\left(-\frac{t}{RC}\right) + CV$$

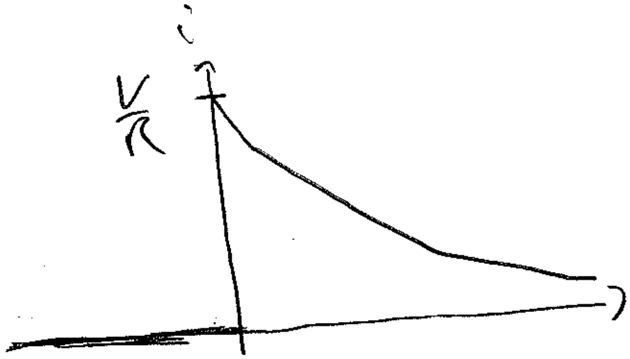
If we have $Q = 0$ at $t = 0$, we can solve for $a \Rightarrow$

$$Q(t) = CV\left[1 - \exp\left(-\frac{t}{RC}\right)\right]$$



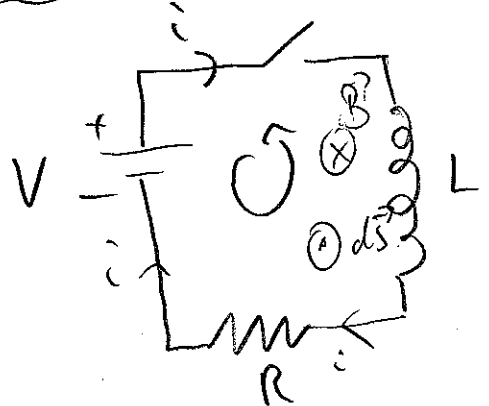
To find i, we can just take the time derivative

$$i(t) = \dot{Q}(t) = \frac{V}{R} \exp\left(-\frac{t}{RC}\right)$$



That the current jumps instantaneously to the value $\frac{V}{R}$ is an artifact of our negligence of the self-inductance of the circuit. It's only a good approximation, if the wires are not too long, because it contradicts relativity which tells us that we can not send signals with a speed larger than the speed of light!

12.3 The RL circuit



This example we have already treated in the previous chapter. Here we note that we set the right equation by Faraday's law again as in the example of the RLC circuit in the beginning of this chapter

$$\oint d\vec{r}\cdot\vec{E} = +V - iR = +L\frac{di}{dt}$$

We found the general solution to be (see p. 107–108)

$$i(t) = \frac{V}{R} + a\exp\left(-\frac{R}{L}t\right)$$

If $i(t=0)=0$ we can solve for a and find

$$i(t) = \frac{V}{R}\left[1-\exp\left(-\frac{R}{L}t\right)\right]$$

12.4. The RC circuit with an AC voltage

Now we can also calculate the case that we hook up some circuit to an AC voltage (as you get from a usual household plug)

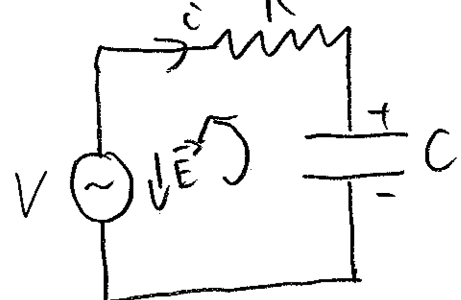
$$V(t) = V_0 \cos(\omega t)$$

NB
For a usual plug, we have

$$V_0 = \sqrt{2} \cdot 120\ V \cong 170\ V$$

and $\omega = 2\pi \cdot \frac{60}{s} \cong 377 \frac{1}{s}$

The RC circuit is treated as before



Ignoring the self-inductance, with the above defined directions of i and $d\vec{r}$, we find

$$\oint d\vec{r} \cdot \vec{E} = V - \frac{Q}{C} - iR = 0$$

and $i = \frac{dQ}{dt}$.

So we have again

$$R\dot{Q} + \frac{Q}{C} = V = V_0 \cos(\omega t)$$

So we can take the solution of the homogeneous solution as before. We look at the current now:

$$i_H = \frac{V}{R} \exp\left(-\frac{t}{RC}\right) \quad ; \quad a = \text{const.}$$

Inhomogeneous solution

Again, we think what should be the state after a long time when the homogeneous solution is damped (i.e. for times $t \gg RC$): Then the generator forces the current (and thus Q) to oscillate with the same frequency as V. However Q needs not to be "in phase" with V. So our ansatz is

$$Q_I(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $A, B = \text{const}$. Then our ODE tells us

$$R\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right] + \frac{1}{C}\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

$$\overset{!}{=} V_0\cos(\omega t)$$

This can be true for all t only if the coefficients in front of $\cos(\omega t)$ and $\sin(\omega t)$ are the same on both sides of the equation. So we find

$$-RA\omega + \frac{B}{C} = 0 \qquad (1)$$

$$R\omega B + \frac{A}{C} = V_0 \qquad (2)$$

$$(1) \Rightarrow A = \frac{B}{R\omega C}$$

This in (2) gives

$$R\omega B + \frac{B}{RC^2\omega} = V_0$$

$$B\left[\frac{(R\omega C)^2 + 1}{RC^2\omega}\right] = V_0$$

$$B = \frac{R\omega C^2}{1+(R\omega C)^2} V_0$$

$$A = \frac{B}{R\omega C} = \frac{C}{1+(R\omega C)^2} V_0$$

So after a long time

$$Q_{t\to\infty} = \frac{C V_0}{1+(R\omega C)^2}\left[\cos(\omega t) + R\omega C \sin(\omega t)\right]$$

The current is

$$i_{t\to\infty} = \frac{\omega C V_0}{1+(R\omega C)^2}\left[R\omega C \cos(\omega t) - \sin(\omega t)\right]$$

This we can also write differently, because we can use

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Setting $\alpha = \omega t$ and $\beta = \varphi_0$ we get

$$\cos(\omega t+\varphi_0) = \cos(\omega t)\cos\varphi_0 - \sin(\omega t)\sin\varphi_0$$

and we can write our solution as

$$i_{t\to\infty}(t) = \frac{\omega C V_0}{\sqrt{1+(R\omega C)^2}}\cos(\omega t+\varphi_0)$$

when φ_0 is determined by

$$\cos\varphi_0 = \frac{R\omega C}{\sqrt{1+(R\omega C)^2}}\,; \quad \sin\varphi_0 = \frac{-1}{\sqrt{1+(R\omega C)^2}}$$

Note that such a φ_0 exists, because we have choosen the coefficient such that

$$\sin^2\varphi_0 + \cos^2\varphi_0 = 1$$

as it must be.

Now $\varphi_0 = + \arccos \dfrac{R\omega C}{\sqrt{1+(R\omega C)^2}} \in [0, \frac{\pi}{2}]$

The sign is determined by the sign of $\sin\varphi_0$ and the magnitude by $\cos\varphi_0$. Thus the φ_0 is positive here.

We can write $\varphi_0 = \omega T_0$. Then we have

$$i_{t\to\infty}(t) = \frac{\omega C V_0}{\sqrt{1+(R\omega C)^2}} \cos[\omega(t+T_0)]$$

since $T_0 > 0$ this means that the current is by a time T_0 advanced compared to $V(t)$.

<u>Special case:</u> $R=0$

Then $\cos\varphi_0 = 0$ and thus $\varphi_0 = +\frac{\pi}{2}$ and

$$i_{t\to\infty}(t) = \omega C V_0 \cos(\omega t + \tfrac{\pi}{2})$$
$$= -\omega C V_0 \sin(\omega t)$$

In this case the phase shift is $\frac{\pi}{2}$ and thus the current "advances" V by a quarter of the period

$$T = \frac{1}{f} = \frac{2\pi}{\omega}.$$

12.5 The RL circuit with an AC voltage

We can use again our result from sect. 12.3:

$$L\frac{di}{dt} + Ri = V = V_0 \cos(\omega t)$$

also the homogeneous equation has the same solution

$$i_H(t) = a \exp\left(-\frac{R}{L}t\right) \;; \quad a = \text{const.}$$

Only the inhomogeneous equation has to be solved again.
As in the previous example, a long time after switching on the voltage, we have

$$i_{t\to\infty}(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$L\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right] + R\left[A\cos(\omega t) + B\sin(\omega t)\right] = V_0\cos(\omega t)$$

Comparing coefficients

$$-\omega L A + R B = 0$$
$$\omega L B + R A = V_0$$

Solving the linear equations gives

$$A = \frac{R V_0}{R^2+\omega^2L^2}\,, \quad B = \frac{\omega L V_0}{R^2+\omega^2L^2}$$

$$i_{t\to\infty}(t) = \frac{R V_0}{R^2+\omega^2L^2}\cos(\omega t) + \frac{\omega L V_0}{R^2+\omega^2L^2}\sin(\omega t)$$

In terms of a phase shift

$$i_{t\to\infty}(t) = i_{max} \cos(\omega t + \varphi_0)$$
$$= i_{max}\left[\cos(\omega t)\cos(\varphi_0) - \sin(\omega t)\sin(\varphi_0)\right]$$
$$= \frac{V_0}{\sqrt{R^2+\omega^2L^2}}\left[\frac{R}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t) + \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t)\right]$$

$$\Rightarrow \cos(\varphi_0) = \frac{R}{\sqrt{R^2+\omega^2L^2}} \;;\; \sin\varphi_0 = \frac{-\omega L}{\sqrt{R^2+\omega^2L^2}}$$

Since $\sin\varphi_0 > 0$, we have

$$\varphi_0 = -\arccos\frac{R}{\sqrt{R^2+\omega^2L^2}} \in \left[-\frac{\pi}{2}, 0\right]$$

Since φ_0 is positive we have with

$$T_0 = -\frac{\varphi_0}{\omega} > 0$$

$$i_{t\to\infty}(t) = i_{max}\cos\left[\omega(t-T_0)\right]$$

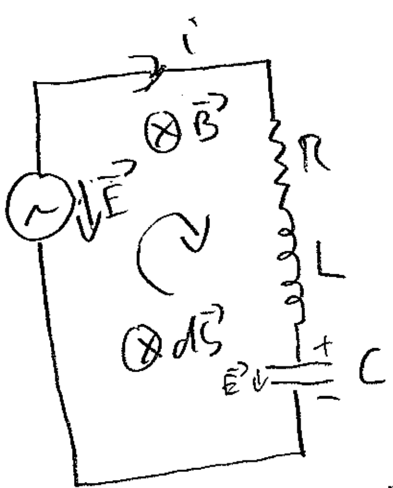
The current is behind of the voltage. That's again due to Lenz's Law: The induction current opposes the change of the current due to the change of voltage.

For $R=0$, $\varphi_0 = -\frac{\pi}{2}$ $\Rightarrow$ The current is quarter a period behind the voltage.

For $L=0$, $\varphi_0 = 0$. Then there is no phase shift.

12.6 Discussion of the RLC circuit with AC voltage

We shall discuss the RLC circuit in detail.



Integrating clockwise in Faraday's Law, we have the following equation

$$\oint d\vec{r}\cdot\vec{E} = Ri + \frac{Q}{C} - V(t) = -L\frac{di}{dt}$$

With the current flowing in the given direction, we have

$$i = \frac{dQ}{dt} \Rightarrow \frac{di}{dt} = \frac{d^2Q}{dt^2}$$

Then our differential equation reads

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t)$$

The general solution is given as the sum of

- a special solution of the equation itself. It does not matter, which solution you choose. We define this solution as $Q_I(t)$ (I = inhomogeneous equation)

- The general solution of the homogeneous equation which is given by setting $V(t) = 0$:

$$L\frac{d^2Q_H}{dt} + R\frac{dQ_H}{dt} + \frac{Q_H}{C} = 0$$

(H = homogeneous equation)

$$\boxed{Q(t) = Q_H(t) + Q_I(t)}$$

12.6.1. General solution for the homogeneous equation

$$L \frac{d^2 Q_H}{dt^2} + R \frac{dQ_H}{dt} + \frac{Q_H}{C} = 0$$

In the calculus course one shows that the most general solution is given by

$$Q_H(t) = A\, Q_H^{(1)}(t) + B\, Q_H^{(2)}(t)$$

where $Q_H^{(1)}$ and $Q_H^{(2)}$ are arbitrary solutions, which must be linearly independent, and A, B are real constants. They are determined by the initial conditions of the full solution:

$$Q(t=0) = Q_0 \;;\; i(t=0) = \left.\frac{dQ(t)}{dt}\right|_{t=0} = i_0$$

To solve the homogeneous equation, we remember that the resistor dissipates energy into heat, and thus we expect the amplitude of the charge to "decay" with time. Thus we make the ansatz

$$Q_H(t) = \exp(-\beta t)\, q(t)$$

where q is a still unknown function. The derivatives read

$$\frac{dQ_H}{dt} = \exp(-\beta t)\left[\frac{dq}{dt} - \beta q\right]$$

$$\frac{d^2 Q_H}{dt^2} = \exp(-\beta t)\left[\frac{d^2 q}{dt^2} - 2\beta \frac{dq}{dt} + \beta^2 q\right]$$

Plugging this in our homogeneous equation yields

$$\exp(-\beta t)\left[L\frac{d^2q}{dt^2}+(R-2L\beta)\frac{dq}{dt}+\left(\frac{1}{C}-\beta+L\beta^2\right)q\right]=0$$

Since $\exp(-\beta t)\neq 0$, we must have

$$L\frac{d^2q}{dt^2}+(R-2L\beta)\frac{dq}{dt}+\left(\frac{1}{C}-\beta+L\beta^2\right)q=0$$

This becomes simpler, if we choose

$$R-2L\beta=0 \Rightarrow \beta=\frac{R}{2L}$$

Then our equation for q becomes

$$\frac{d^2q}{dt^2}=-\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)q$$

To solve this equation we have to distinguish several cases

(a) $\frac{1}{LC}-\frac{R^2}{4L^2}>0$ (oscillatory)

Then we can write

$$\frac{d^2q}{dt^2}=-\omega^2 q \text{ with } \omega=\sqrt{\frac{1}{LC}-\frac{R^2}{4L}}\in\mathbb{R}.$$

This is the equation of the harmonic oscillator, and we know the general solution

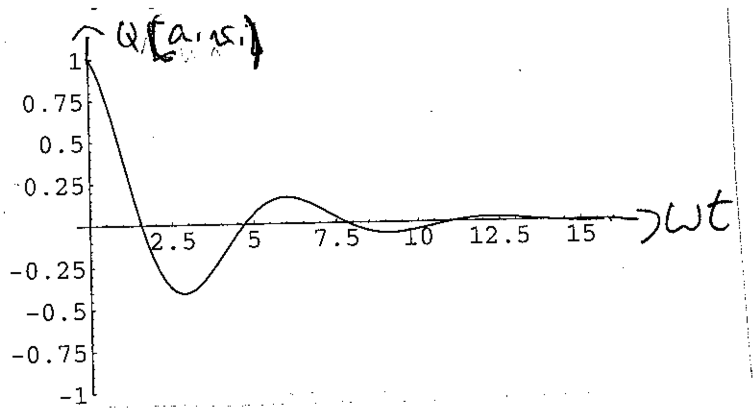
$$q(t)=A\cos(\omega t)+B\sin(\omega t).$$

Since $\frac{\sin(\omega t)}{\cos(\omega t)}=\tan(\omega t)$ is <u>not</u> constant, we have

indeed the most general solution!
In this case, we have the solution for our original homog. equation

$$Q_H(t) = \exp(-\beta t)\, q(t)$$

$$\boxed{\Rightarrow Q_H(t) = \exp(-\beta t)\left[A\cos(\omega t) + B\sin(\omega t)\right] \quad \text{with } \beta = \frac{R}{2L}\,;\ \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$



This shows that we have a damped oscillatory solution for the equation. The solution becomes very small for times

$$t \gg \frac{1}{\beta} = \frac{2L}{R}$$

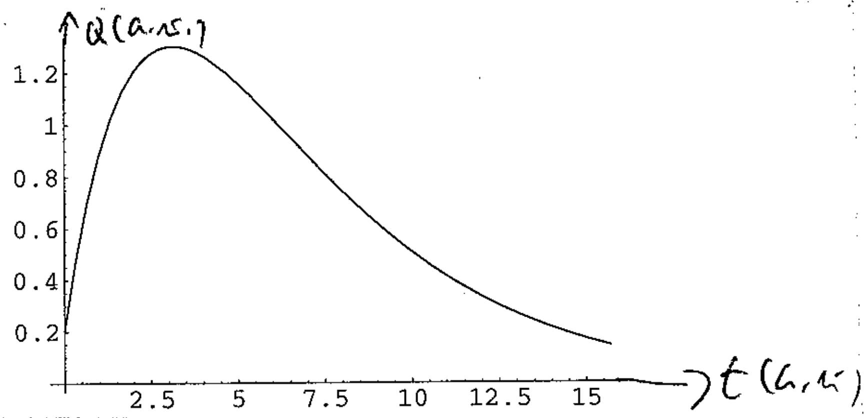
(b) $\frac{1}{LC} - \frac{R^2}{4L^2} = 0$ (aperiodic limit)

$$\Rightarrow \frac{d^2q}{dt^2} = 0 \Rightarrow q(t) = At + B$$

($\frac{At}{B}$ is not constant $\Rightarrow$ solution complete!)

The original equation has thus the solution

$$\boxed{Q(t) = \exp(-\beta t)\,(At+B) \quad \text{with } \beta = \frac{R}{2L}}$$



(c) $\frac{1}{LC} - \frac{R^2}{4L^2} < 0$ (overdamping)

Then we have

$$\frac{d^2q}{dt^2} = \lambda^2 q \quad \text{with } \lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \in \mathbb{R}$$

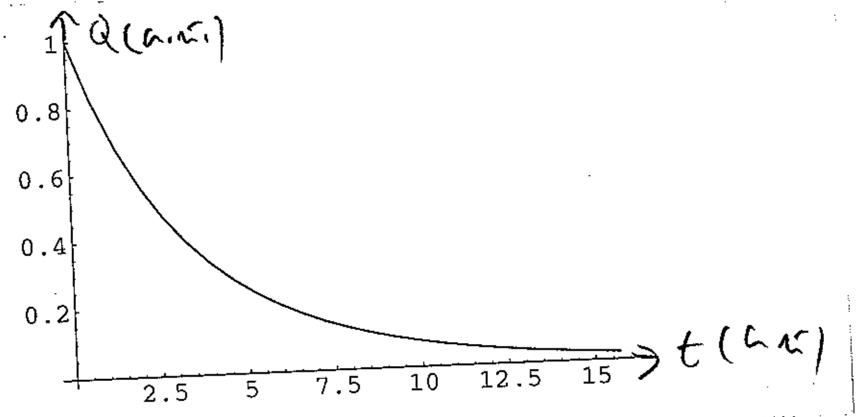
The general solution for this is

$$q(t) = A\exp(\lambda t) + B\exp(-\lambda t)$$

The solution for the original equation thus is

$$\boxed{\begin{aligned} &Q(t) = A\exp(-\lambda_1 t) + B\exp(-\lambda_2 t) \\ &\text{with } \left.\begin{aligned}\lambda_1 &= \frac{R}{2L} - \lambda \\ \lambda_2 &= \frac{R}{2L} + \lambda\end{aligned}\right\} \lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\end{aligned}}$$

Since $\lambda < \frac{R}{2L}$ the solution is damped as it must be. The charge cannot grow indefinitely on the capacitor if there is no source



12.6.2 Particular solution for the inhomogeneous equation

As we have seen, the solutions of the homogeneous equation are damped after a long time. With the AC voltage

$$V(t) = V_0 \cos(\omega_0 t)$$

after the homogeneous part is damped away we should have an oscillatory solution with the same frequency:

$$Q(t) = a \cos(\omega_0 t) + b \sin(\omega_0 t)$$

Plugging this into the equation

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V_0 \cos(\omega_0 t)$$

we find

$$\frac{1}{C}\Big[[a(1 - LC\omega_0^2) + bRC\omega_0] \cos(\omega_0 t) +$$

$$+ [b(1 - LC\omega_0^2) - aRC\omega_0] \sin(\omega_0 t) \Big] =$$

$$= V_0 \cos(\omega_0 t)$$

It follows

$$a\,(1-LC\omega_0^2) + b\,RC\omega_0 = CV_0$$
$$b\,(1-LC\omega_0^2) - a\,RC\omega_0 = 0$$

The solution for this set of linear equations is

$$a = \frac{C V_0 (1-LC\omega_0^2)}{R^2C^2\omega_0^2 + (1-LC\omega_0^2)^2}$$

$$b = \frac{C^2 R \omega_0 V_0}{R^2C^2\omega_0^2 + (1-LC\omega_0^2)^2}$$

Now we write this in terms of amplitude and phase shift

$$Q = \hat{Q} \cos(\omega_0 t + \varphi_Q)$$
$$= \hat{Q}\,[\cos\varphi_Q \cos(\omega_0 t) - \sin\varphi_Q \sin(\omega_0 t)]$$

$$\Rightarrow \quad \left.\begin{array}{l} a = \hat{Q}\cos\varphi_Q \\ b = -\hat{Q}\sin\varphi_Q \end{array}\right\} \Rightarrow \hat{Q} = \sqrt{a^2+b^2} = \frac{V_0 C}{\sqrt{(RC\omega_0)^2 + (1-LC\omega_0^2)^2}}$$

$$\cos\varphi_Q = \frac{a}{\hat{Q}} = \frac{1-LC\omega_0^2}{\sqrt{(RC\omega_0)^2 + (1-LC\omega_0^2)^2}}$$

$$\sin\varphi_Q = -\frac{b}{\hat{Q}} = \frac{-RC\omega_0}{\sqrt{(RC\omega_0)^2 + (1-LC\omega_0^2)^2}} \leq 0$$

So we have

$$\varphi_Q = -\arccos\left(\frac{1-LC\omega_0^2}{\sqrt{(RC\omega_0)^2 + (1-LC\omega_0^2)^2}}\right) \in [-\pi, 0]$$

or

$$\varphi_Q = -\arccos\left(\frac{\frac{1}{\omega_0 C} - \omega_0 L}{\sqrt{R^2 + \left(\frac{1}{\omega_0 C} - L\omega_0\right)^2}}\right) \in [-\pi, 0]$$

One also writes

$$\hat{Q} = \frac{V_0}{\omega_0 \sqrt{R^2 + \left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}}$$

Then one obtains for the current

$$i(t) = \frac{dQ}{dt} = -\frac{V_0}{\sqrt{R^2+\left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}} \sin(\omega_0 t + \varphi_Q)$$

$$\boxed{i(t) = \frac{V_0}{\sqrt{R^2+\left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}} \cos\left(\omega_0 t + \varphi_Q + \frac{\pi}{2}\right)}$$

$$i(t) = \hat{i} \cos(\omega_0 t + \varphi_0)$$

with $\varphi_0 = \varphi_Q + \frac{\pi}{2} \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

At a frequency $\omega_0 = \omega_R$, for which

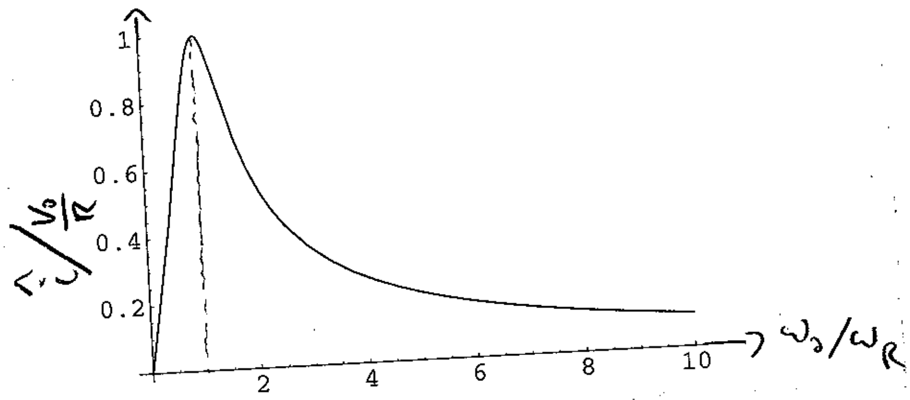
$$\frac{1}{\omega_R C} - \omega_R L = 0 \Rightarrow \omega_R = \sqrt{\frac{1}{LC}}$$

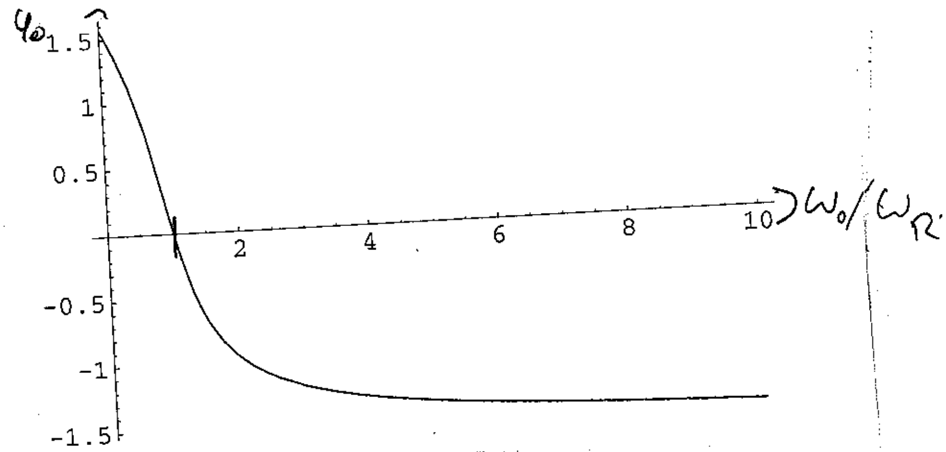
the amplitude for the current becomes maximal. That's called a resonance:

$$\hat{i}_R = \frac{V_0}{R}$$

The phase shift between current and voltage then is

$$\varphi_0^{(R)} = \varphi_Q^{(R)} + \frac{\pi}{2} = 0, \text{ because } \varphi_Q^{(R)} = -\frac{\pi}{2}$$





For $\omega_0 = \omega_R$ the phaseshift between current and voltage is 0, i.e., the current follows precisely the voltage. Current and voltage are "in phase".

For $\omega_0 < \omega_R$ we have $\varphi_0 < 0$. That means that the current advances the voltage. The formula shows that for low frequencies the capacitor dominates the behavior. That's why $\varphi_0 < 0$.

For $\omega_0 > \omega_R$ we find $\varphi_0 > 0$, i.e., the current stays behind the voltage. Here the self-inductance of the coil dominates and works against the changes of the currents enforced by the change of the voltage.

Average power of an AC circuit

As we have seen on several examples, we can describe the stationary state of an AC circuit with a single voltage with frequency $f = \frac{\omega}{2\pi}$ by

$$V(t) = V_0 \cos(\omega t)$$
$$i(t) = i_0 \cos(\omega t + \varphi_0)$$

The momentary power used by the circuit is

$$P(t) = V(t)\, i(t) = V_0 i_0 \cos(\omega t) \left[\cos(\omega t)\cos(\varphi_0) - \sin(\omega t)\sin(\varphi_0)\right]$$

The average Power is defined by

$$\langle P \rangle = \frac{1}{T} \int_0 dt\, P(t)$$

where $T = \frac{2\pi}{\omega}$ is the period.

Now

$$\cos(2\omega t) = \cos^2(\omega t) - \sin^2(\omega t)$$
$$= \cos^2(\omega t) - [1 - \cos^2(\omega t)]$$
$$= 2\cos^2(\omega t) - 1$$

$$\Rightarrow \quad \cos^2(\omega t) = \frac{1}{2}[1 + \cos(2\omega t)]$$

$$\sin(2\omega t) = 2\sin(\omega t)\cos(\omega t)$$

$$\Rightarrow \quad \sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$$

$$\Rightarrow \int_0^T dt \cos^2(\omega t) = \int_0^T dt\, \frac{1}{2}[1 + \cos(2\omega t)]$$
$$= \frac{T}{2} + \frac{1}{4\omega}\sin(2\omega t)\Big|_0^T = \frac{T}{2}$$

$$\int_0^T dt \cos(\omega t)\sin(\omega t) = \frac{1}{2}\int_0^T dt \sin(2\omega t)$$

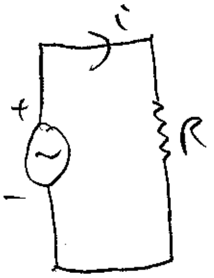
$$= -\frac{1}{4\omega}\cos(2\omega t)\Big|_0^T = 0$$

Thus

$$\boxed{\langle P\rangle = \frac{1}{2} V_0 i_0 \cos\varphi_0}$$

Effective voltage and current for an AC circuit

The effective voltage and current of an AC source is defined by the values for a DC source which delivers the same power at a purely Ohmian resistor



From Faraday's Law, neglecting L, we get

$$R\, i(t) = V(t) = V_0 \cos(\omega t)$$

$$\Rightarrow i(t) = \frac{V_0}{R}\cos(\omega t)$$

This means that $i_0 = \frac{V_0}{R}$ and $\varphi_0 = 0$. Thus the average power is

$$\langle P\rangle = \frac{1}{2} V_0 i_0 = \frac{R^2}{2} i_0^2 = \frac{V_0^2}{2R}$$

For a DC source we would get

$$P = R^2 i_{eff}^2 = \frac{V_{eff}^2}{2R} \stackrel{!}{=} \langle P\rangle$$

So: $\quad i_{eff}^2 = \frac{1}{2} i_0^2 \Rightarrow i_{eff} = \frac{i_0}{\sqrt{2}}$

and $\quad V_{eff}^2 = \frac{1}{2} U_0^2 \Rightarrow V_{eff} = \frac{U_0}{\sqrt{2}}$.

These are the values given on AC sources (like 110 V for a usual household plug).