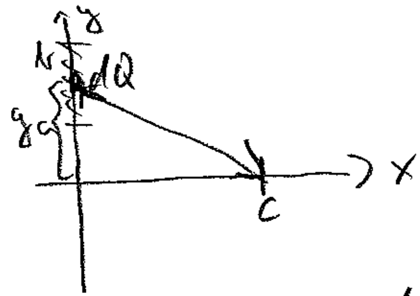
Exam I



1st solution: direct calculation

$$\vec{E} = \int \frac{dQ}{4\pi\varepsilon_0} \frac{1}{\sqrt{y^2+c^2}^{\,2}} \frac{+c\,\vec{i}_x - y\,\vec{i}_y}{\sqrt{y^2+c^2}}$$

$$= \int_a^b dy \frac{Q}{(b-a)4\pi\varepsilon_0} \frac{(c\,\vec{i}_x - y\,\vec{i}_y)}{(y^2+c^2)^{3/2}}$$

$$E_x = \int_a^b dy \frac{Qc}{(b-a)4\pi\varepsilon_0} \frac{1}{(y^2+c^2)^{3/2}}$$

Substitution:

$$y = c\sinh u$$

$$dy = c\cosh u\, du$$

$$\int dy (y^2+c^2)^{-3/2} = \int du\, c\cosh u \frac{\cosh^{-3} u}{c^3}$$

$$= \frac{1}{c^2}\int \frac{du}{\cosh^2 u} = \frac{1}{c^2}\tanh u = \frac{1}{c^2}\frac{\sinh u}{\cosh u}$$

$$= \frac{1}{c^2}\frac{y/c}{\sqrt{1+\frac{y^2}{c^2}}} = \frac{y}{c^2\sqrt{c^2+y^2}}$$

$$E_x = \frac{Q}{c(b-a)} \frac{1}{4\pi\varepsilon_0} \left[\frac{b}{\sqrt{c^2+b^2}} - \frac{a}{\sqrt{c^2+a^2}} \right]$$

$$E_y = -\int_a^b dy \, \frac{Q\, y}{(b-a)\, 4\pi\varepsilon_0} \frac{1}{(c^2+y^2)^{3/2}}$$

$$= - \frac{Q}{4\pi\varepsilon_0 (b-a)} \left(- \frac{1}{\sqrt{c^2+y^2}} \right)_a^b$$

$$= + \frac{Q}{4\pi\varepsilon_0 (b-a)} \left[\frac{1}{\sqrt{c^2+b^2}} - \frac{1}{\sqrt{c^2+a^2}} \right]$$

2^{nd} solution: via potential

Here we need all points

$$V = \frac{1}{4\pi\varepsilon_0} \int_a^b dy' \, \frac{Q}{b-a} \frac{1}{\sqrt{x^2+(y-y')^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{b-a} \ln \left(\frac{y-a+\sqrt{x^2+(y-a)^2}}{y-b+\sqrt{x^2+(y-b)^2}} \right)$$

$$4\pi\varepsilon_0 E_x = - \frac{\partial V}{\partial x} = - \frac{Q}{b-a} \left[\frac{x}{\sqrt{x^2+(y-a)^2}\,(y-a+\sqrt{x^2+(y-a)^2})} - \frac{x}{\sqrt{x^2+(y-b)^2}\,(y-b+\sqrt{x^2+(y-b)^2})} \right]$$

For $x=c$, $y=0$

$$4\pi\varepsilon_0 E_x = -\frac{Q}{b-a}\left[\frac{c}{\sqrt{c^2+a^2}\left(-a+\sqrt{c^2+a^2}\right)}\right.$$

$$\left. - \frac{c}{\sqrt{c^2+b^2}\left(-b+\sqrt{c^2+b^2}\right)}\right)$$

$$= -\frac{Q}{b-a}\left[\frac{c}{\sqrt{c^2+a^2}}\frac{\left(a+\sqrt{c^2+a^2}\right)}{c^2} - \frac{c}{\sqrt{c^2+b^2}}\cdot\frac{b+\sqrt{c^2+b^2}}{c^2}\right]$$

$$= -\frac{Q}{b-a}\left[\frac{a}{c\sqrt{c^2+a^2}} - \frac{b}{c\sqrt{c^2+b^2}}\right]$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{Q}{b-a}\left[\frac{1}{\sqrt{c^2+b^2}} - \frac{1}{\sqrt{c^2+a^2}}\right]$$

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(2) $$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|^3}$$

$$\vec{F}_{32} = \frac{(-q_3)q_2}{4\pi\varepsilon_0}\frac{\vec{r}_2-\vec{r}_3}{|\vec{r}_2-\vec{r}_3|^3}$$

$$\vec{F}_{all_{12}} = \vec{F}_{12}+\vec{F}_{32} = \frac{q_2}{4\pi\varepsilon_0}\left(q_1\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|^3} - q_3\frac{\vec{r}_2-\vec{r}_3}{|\vec{r}_2-\vec{r}_3|^3}\right)$$

(3)

$$d\vec{S} = A^2 \sin\vartheta \, d\vartheta \, d\varphi \, \vec{i}_r$$

$$\frac{\Phi_A}{\varepsilon_0} = \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \, A^2 \sin\vartheta \, d\vartheta \, d\varphi \, \alpha A^3$$

$$= \alpha A^5 \, 2\pi \int_0^{\pi} d\vartheta \sin\vartheta = 4\pi \alpha A^5$$

check via local law:

$$\vec{E} = \alpha r^3 \vec{i}_r$$

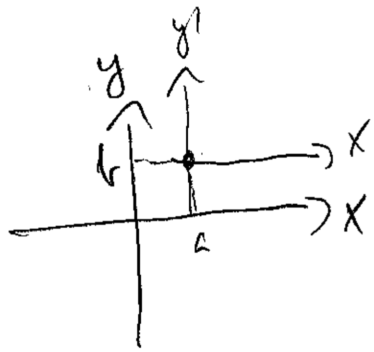
$$\mathrm{div}\, \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (\alpha r^5) = 5 \alpha r^2$$

$$\frac{\Phi_A}{\varepsilon_0} = \int_V dV \, \mathrm{div}\, \vec{E} = \int_0^A r^2 dr \int_0^{\pi} d\vartheta \int_0^{\pi} d\varphi \; 5\alpha r^2 \sin\vartheta$$

$$= 4\pi \int_0^A dr \; 5 \alpha r^4$$

$$= \underline{\underline{4\pi \alpha A^5}}$$

(4)



For charge at origin

$$V_{Q@0} \Rightarrow V_{Q_0} = V_{Q_0}(r') \; ; \; r' = \sqrt{x'^2 + y'^2}$$

$$\vec{E} = \gamma \frac{Q}{r'^4} \vec{i}_{r'}$$

$$= -\frac{\partial V}{\partial r} \vec{i}_{r'}$$

$$\Rightarrow -\frac{\partial V}{\partial r} = \gamma \frac{Q}{r'^4}$$

$$\Rightarrow \quad \frac{\partial V}{\partial r} = -\gamma Q r'^{-4}$$

$$V = \frac{1}{3} \gamma Q r'^{-3}$$

Soma $x' = x - a$, $y' = y - b$

$$\boxed{V = \frac{\gamma Q}{3} \frac{1}{[(x-a)^2 + (y-b)^2]^{3/2}}}$$