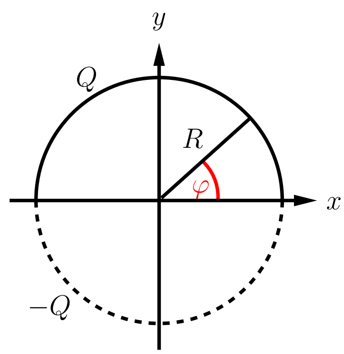
Name: .... Section:.................

Physics 208 Quiz 2

Solutions

January 25, 2008 (due: February 1, 2008)

Problem 1 (50 points)



A circle (radius, R) is located in the center of a Cartesian coordinate system (see figure). The upper half is uniformly charged with charge, Q, the lower half with charge, $-Q$

(a) Calculate the electric field, $\vec{E}$, at the center of the circle! I parametrize the circle with the angle, φ, as in the figure. Then, the charge per angle is

$$\frac{\mathrm{d}Q(\varphi)}{\mathrm{d}\varphi} = \lambda(\varphi) = \begin{cases} \frac{Q}{\pi} & \text{for } 0 < \varphi < \pi \\ -\frac{Q}{\pi} & \text{for } \pi < \varphi < 2\pi \end{cases} \tag{1}$$

According to Coulomb's Law the electric field in the origin is given by

$$\vec{E}(0) = -\int_0^{2\pi} \mathrm{d}\varphi (\cos\varphi \vec{i}_x + \sin\varphi \vec{i}_y) \frac{\lambda(\varphi)}{4\pi\epsilon_0 R^2}. \tag{2}$$

Then we have to calculate the integral from 0 to π and then the one from π to 2π separately, because of the form of (1). With help of the integrals $\int \mathrm{d}\varphi \cos\varphi = \sin\varphi$ and $\int \mathrm{d}\varphi \sin\varphi = -\cos\varphi$, we finally obtain

$$\vec{E}(0) = -\frac{Q}{\pi^2 \epsilon_0 R^2} \vec{i}_y. \tag{3}$$

(b) What is the force on a test charge, q_0, located in the center?

$$\vec{F}(0) = q_0 \vec{E}(0) = -\frac{q_0 Q}{\pi^2 \epsilon_0 R^2} \vec{i}_y. \tag{4}$$

Problem 2 (50 points)

A particle with mass, m, and charge, $q < 0$, moves in the field of a point charge, $Q > 0$, which is fixed in the origin of a Cartesian coordinate system. The particle starts at rest in a distance, R, on the x axis of a Cartesian coordinate system: $\vec{r}_0 = R\vec{i}_x$, $\vec{v}_0 = 0$.

(a) What is the force, acting on the particle with charge, q. Write down its equation of motion:

$$m\vec{a} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \vec{F}(\vec{r}).$$

From Coulomb's Law

$$\vec{F} = \frac{Qq}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} := -A\frac{\vec{r}}{r^3}, \text{ where } A = -\frac{Qq}{4\pi\epsilon_0} > 0. \tag{5}$$

The equation of motion (EoM) reads

$$m\vec{a} = m\ddot{\vec{r}} = -A\frac{\vec{r}}{r^3}. \tag{6}$$

(b) Show that the particle moves in a straight line along the x axis!

From the initial conditions $\vec{r}_0 = R\vec{i}_x$, $\vec{v}_0 = 0$ it is clear that the particle starts at rest on the x axis. In the first instance the force acts in direction of the x axis towards the center, and the particle starts to move along the x axis. The force still acts along the x axis. Thus, we conclude that the particle just runs towards the center along the x axis. To prove this formaly, we make the ansatz $\vec{r}(t) = x(t)\vec{i}_x$. Then, writing the EoM in terms of its components, we get

$$\begin{aligned} m\ddot{x} &= -\frac{Ax}{|x|^3}, \\ m\ddot{y} &= 0, \\ m\ddot{z} &= 0. \end{aligned} \tag{7}$$

Obviously, this is solved with $y(t) = 0$ and $z(t) = 0$, fulfilling the initial conditions.

(c) Prove the energy-conservation law:

$$\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{qQ}{4\pi\epsilon_0|x|} = E = \text{const}. \tag{8}$$

What is E (in terms of m, q, Q, and R)?

Hint: Take the time derivative of the expression above and use the equation of motion for x to show that it vanishes, i.e., $\mathrm{d}E/\mathrm{d}t = 0$. To find E, plug the initial condition into Eq. (8). For simplification, you can assume that $x > 0$.

As said in the hint, we assume $x > 0$. Then we can write

$$E = \frac{m}{2}v_x^2 - \frac{A}{x} \tag{9}$$

The time derivative of this expression is (product rule!)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\dot{x}\ddot{x} + \frac{A\dot{x}}{x^2} = \dot{x}\left(m\ddot{x} + \frac{A}{x^2}\right) \tag{10}$$

For $x > 0$, $|x| = x$, and from (7) we see that the bracket vanishes, i.e., $\mathrm{d}E/\mathrm{d}t = 0$ and thus $E = \text{const}$. With the initial condition we find

$$E = -\frac{A}{R}. \tag{11}$$

(d) (for extra credit): How long does it take for the particle to reach the center (where the charge, Q, sits).

Hint: use the energy-conservation law from part (c)! You can use the following integral:

$$\int_0^R \mathrm{d}x\sqrt{\frac{x}{R-x}} = \frac{\pi R}{2}.$$

That's indeed a little bit tricky: it is clear that we can solve (9) for v_x. We only have to be careful with the sign when taking the square root. Qualitatively it is clear that the particle moves from positive x towards the center, i.e., $x = 0$. So the velocity-x component is negative. Solving (9) for v_x thus gives

$$v_x = \dot{x} = -\sqrt{\frac{2A}{mR}}\sqrt{\frac{R-x}{x}}. \tag{12}$$

This is of course only a sensible (i.e., real) expression, if $0 < x \leq R$, but that's fine for our problem, since we want to know only the time, until the particle runs into the center. To find this time, we remember that the derivative of the inverse function is given by the inverse of the derivative of the original function, i.e.

$$\frac{1}{v_x} = \frac{\mathrm{d}t}{\mathrm{d}x} = -\sqrt{\frac{mR}{2A}}\sqrt{\frac{x}{R-x}}. \tag{13}$$

Now we have to integrate both sides from $x = R$ to $x = 0$, where $t = 0$ and $t = T > 0$, respectively (T is the searched time it takes the particle to hit the center). Switching the boundaries on the right-hand side and the sign, this gives finally

$$T = \sqrt{\frac{mR}{2A}} \int_0^R \mathrm{d}x \sqrt{\frac{x}{R-x}} = \sqrt{\frac{mR}{2A}} \frac{\pi R}{2}, \tag{14}$$

where we have used the integral, given in the hint.

Since this part is **really** tricky, you get the full score (100 points) without solving problem (d), but you could earn up to 10 points extra credit!