Name: .... Section: ....

Physics 208 Quiz 6

March 26, 2008; due April 4, 2008

Problem 1 (50 points)

In the lecture (on March 24) we have seen that a magnetic dipole field can be described by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{r}\vec{p}_m)\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3}\right].$$

Here, $\mu_0 = \text{const}$ is a constant, we will learn about in detail in the next chapter; $\vec{p}_m$ is a constant vector called the magnetic dipole moment, and $\vec{r}$ is the position vector of an arbitrary point.

Let S be a sphere with radius R around the origin. Show that there is indeed no "magnetic charge" present as it should be, i.e.,

$$\int_S \mathrm{d}\vec{S}\vec{B} = 0.$$

Hint: Is is helpful to choose appropriate coordinates, namely spherical coordinates with the polar axis (in our standard notation that is the z axis) in the same direction as $\vec{p}_m$.

Solution

As indicated in the question we choose the z axis of our Cartesian coordinate system along the magnetic dipole moment:

$$\vec{p}_m = p_m \vec{i}_z \tag{1}$$

and introduce usual spherical coordinates

$$\vec{r} = r\cos\varphi\,\sin\vartheta\,\vec{i}_x + r\sin\varphi\,\sin\vartheta\,\vec{i}_y + r\cos\vartheta\,\vec{i}_z. \tag{2}$$

The sphere around the origin with radius R has the surface-element vector (see the math sheet about coordinates on the course webpage!)

$$\mathrm{d}\vec{S} = R^2\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\vec{i}_r. \tag{3}$$

Further, from (1-2) we find along the sphere ($r = R$):

$$\vec{p}_m\vec{r} = p_m R\cos\vartheta, \quad \vec{p}_m\vec{i}_r = p_m\cos\vartheta. \tag{4}$$

Using the diplole field given in the question we find

$$\begin{aligned}\Phi_B = \int_S \mathrm{d}\vec{S}\,\vec{B} &= \frac{\mu_0}{4\pi}\int_0^{\pi}\mathrm{d}\vartheta\int_0^{2\pi}\mathrm{d}\varphi R^2\sin\vartheta\left[\frac{3R^2p_m\cos\vartheta}{R^5} - \frac{p_m\cos\vartheta}{R^3}\right]\\ &= \frac{\mu_0 p_m}{2\pi R}\int_0^{\pi}\mathrm{d}\vartheta\int_0^{2\pi}\mathrm{d}\varphi\sin\vartheta\cos\vartheta.\end{aligned} \tag{5}$$

Since the integrand does not depend on φ, the integral over φ gives just a factor 2π, and we are left with the integral

$$I = \int_0^\pi \mathrm{d}\vartheta \, \sin\vartheta \, \cos\vartheta. \tag{6}$$

To find the integral we substitute $u = \cos\vartheta$, leading to $\mathrm{d}\vartheta \, \sin\vartheta = -\mathrm{d}u$ and thus

$$I = + \int_{-1}^{1} \mathrm{d}u \, u = 0. \tag{7}$$

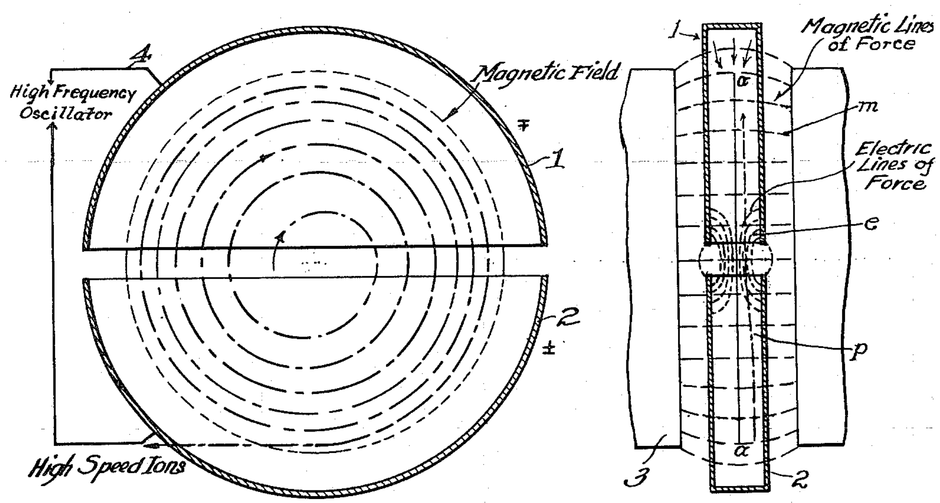
So we find that indeed $\Phi_B = 0$ as it should be. There are no magnetic charges ("monopoles") present. Up to now all searches for magnetic monopoles have failed. Thus we assume that there are none, i.e., for each closed surface S

$$\oint \mathrm{d}\vec{S} \, \vec{B} = 0. \tag{8}$$

This is one of the basic equations of electromagnetism, known as **Maxwell's equations**. As we will learn later, it is generally valid, not only for the static case we have considered so far.

Problem 2 (50 points)

In the early 1930ies, Ernest Lawrence invented a new type of particle accelerator called Cyclotron. To understand the principle, we show a figure contained in Lawrence's patent application:



The apparatus consists of two D-shaped electrodes (called simply D's by accelerator physicists). They are connected to a radio-frequency (RF) voltage of frequency f. The voltage at the electrodes is thus changing with time by

$$V(t) = V_{\max} \cos(\omega t), \quad \omega = 2\pi f.$$

Now by some mechanism, at $t = 0$ a proton (charge $q = 1.6 \cdot 10^{-19}$ C, mass $m = 1.67 \cdot 10^{-27}$ kg) is produced in the gap between the poles. This is the beginning of the dashed line, denoted "high speed ions" in the picture. The voltage is such that the lower plate at this moment is the + terminal, and the upper plate the − terminal (i.e., the upper signs indicated in the figure). Thus the proton becomes accelerated upwards as shown in the left part of the figure.

The whole apparatus is evacuated (such that the protons suffer no friction with air), and a magnetic field, $\vec{B}$, perpendicular to the plane pointing outwards, is applied. As we know from the lecture, this forces the protons on a circle. After half a cycle, the protons enter again the gap between the poles, and the magnitude of the B field is chosen such that now the electric field is pointed precisely in the other direction (indicated by the lower signs in the figure) with a voltage difference V_{max}. Thus, again the protons gain the maximal possible acceleration. This condition is called the "resonance condition".

With the knowledge from the lecture, you can easily calculate Lawrence's original setup (which, in 1939 earned Lawrence the Nobel Prize in physics!).

1. It is given that the magnetic field in Lawrence's apparatus in one case has been of magnitude $B = 0.693\ \text{T} = 0.693\ \text{Wb/m}^2$ and that Lawrence has chosen the frequency of the voltage such that the protons cycle around exactly once in one period of the RF voltage. What is this frequency, $f = \omega/(2\pi)$, of the voltage, Lawrence has used in this case? Explain briefly, why this is a good choice, meeting the "resonance condition", explained above.

2. The radius of the apparatus, within which the principle works, has been about $r_{\text{max}} = 28$ cm. The maximum voltage of the RF generator used has been $V_{\text{max}} = 4000$ V. Calculate the maximal energy of the protons, that Lawrence could reach with his apparatus. *Hint:* Note that the protons are accelerated twice per cycle and, in the ideal case assumed here, in each acceleration run through the full voltage difference, V_{max}.

3. How many cycles have the protons made to full acceleration?

4. (*for extra credit*): Suppose, you want to build a cyclotron, but you cannot reach as high magnetic fields as Lawrence could. Which are smaller allowed values for B to meet the "resonance condition", i.e., that the protons are always accelerated by the maximum available voltage, V_{max}, when they run through the gap of the poles? What is the disadvantage?

Hint: You find the Lawrence's original paper in the Physical Review:
E. O. Lawrence, M. S. Livingston, Phys. Rev. **40** (1932) 19.

It is available online (if you use an internet connection within the university) at the following URL:

http://link.aps.org/abstract/PR/v40/p19

If you read the paper, note that Lawrence uses different units, called Gaussian CGS units, than we do!

Solution

Ad 1. In the lecture we have solved the equations of motion for a charged particle in a constant magnetic field, $\vec{B} = B_z \vec{i}_z$. We have found out that it goes in a circle in the plane perpendicular to the magnetic field, provided $v_z = 0$ which we assume to be true in the following. The solution also gave this angular velocity to be

$$\omega_p = \frac{qB}{m}. \tag{9}$$

Lawrence used the remarkable fact that this is independent of the radius of the particle's circular orbit. That means, if the proton becomes faster, the radius must become larger by the same amount so that it always takes the same time

$$T_p = \frac{2\pi}{\omega_p} \tag{10}$$

to run through the full circle. With the given field strength, mass and charge of the proton we find for the particular example from Lawrence's paper

$$f_p = \frac{\omega_p}{2\pi} = 10.57 \cdot 10^6 \, \frac{1}{\text{s}} = 10.57 \text{ MHz}. \tag{11}$$

As indicated in the question, Lawrence chose this radio frequency (RF) for the AC voltage, i.e., $f = \omega/(2\pi) = f_p$. This is a good choice since in the ideal case then the proton always runs through the full voltage difference, $V_{\text{max}} = 4$ kV.

To understand this, suppose, it starts at $t = 0$ at the lower terminal in the gap. Then it is accelerated upwards towards the upper terminal. There it is bent by the magnetic field, and inside the D-shaped cavity there is no electric field and thus keeps its speed gained before running through the gap. So it runs half a circle which takes the time $T_p/2$. But that is also half a cycle of the RF voltage and thus now the upper terminal is positively charged and the proton is again accelerated running through the full voltage drop, V_{max}, inside the gap.

20 points

Ad 2. The velocity of the proton is related to the angular velocity by

$$v = r\omega_p = \frac{rqB}{m} \tag{12}$$

where r is the radius. The maximal velocity, you can reach with the cyclotron is thus reached when $r = r_{\text{max}}$:

$$v_{\text{max}} = r_{\text{max}}\omega_p = 18.6 \cdot 10^6 \, \frac{\text{m}}{\text{s}}. \tag{13}$$

The maximal reachable energy is thus

$$E_{\text{max}} = \frac{m}{2} v_{\text{max}}^2 = 2.89 \cdot 10^{-13} \text{ J} = 1.8 \text{ MeV}. \tag{14}$$

Here, we introduced a unit for energy which is very useful in particle physics, the *electron volt*. One electron volt is the energy an electron (or any particle with 1 elementary unit of charge, $e = 1.6 \cdot 10^{-19}$ C) gains when it runs through a voltage difference of 1 V, i.e.,

$$1 \text{ eV} = 1.6 \cdot 10^{-19} \text{ J}. \tag{15}$$

The eV is a special unit allowed in the international system of units (for the use in particle physics). So it is also allowed to use the usual prefixes like "kilo" (k) "mega" (M), etc. Thus 1 MeV = 10^6 eV.

20 points

ad 3. In each cycle the protons run through the gap twice and thus, in the here assumed ideal case, gain the energy

$$\Delta E = 2qV = 8 \text{ keV}. \tag{16}$$

So, the number of cycles to reach the maximal energy is

$$N_{\text{cycles}} = \frac{E_{\text{max}}}{\Delta E} = 225. \tag{17}$$

As one can read in the above mentioned paper, Lawrence reached protons with an average energy of 1.22 MeV. Our estimate is thus in the right order of magnitude, and of course in reality not all protons are perfectly synchronized as we assumed for our ideal case.

10 points

ad 4. According to (9), with a smaller magnetic field you can only get smaller angular velocities of the protons, ω_p. This gives you of course a longer time, T_p, to run through a full cycle. To meet the resonance condition, your T_p must be an odd multiple of the RF's time perios, i.e., you must have

$$T_p' = (2n+1)T = \frac{2n+1}{f}, \quad n \in \{0, 1, 2, \ldots\}. \tag{18}$$

Lawrence had a magnetic field B to reach the shortest possible time for the given RF, f (corresponding to $n = 0$). Since you have not as strong a magnet, you have to choose a longer time. The good values for your B field would thus be

$$B' = \frac{B}{2n+1}, \quad n \in \{1, 2, \ldots\} \tag{19}$$

The disadvantage then is that due to your smaller angular velocity,

$$\omega_p' = \frac{\omega_p}{2n+1}, \tag{20}$$

with a given radius r_{max} of your D's you can reach only smaller velocities

$$v_{\text{max}}' = r_{\text{max}} \omega_p' = \frac{v_{\text{max}}}{2n+1}, \tag{21}$$

and thus finally smaller energies:

$$E_{\text{max}}' = \frac{m}{2} {v_{\text{max}}'}^2 = \frac{E_{\text{max}}}{(2n+1)^2}. \tag{22}$$

To put it in another way, to reach the same energies as Lawrence you would have to make the radius by a factor $2n + 1$ larger. This would cause another serious trouble: To make the cyclotron principle work, you need a magnetic field which is constant throughout the whole region to a high accuracy, and this is not so easily achieved for larger regions (see Lawrence's paper for details about this issue).

10 points extra credit