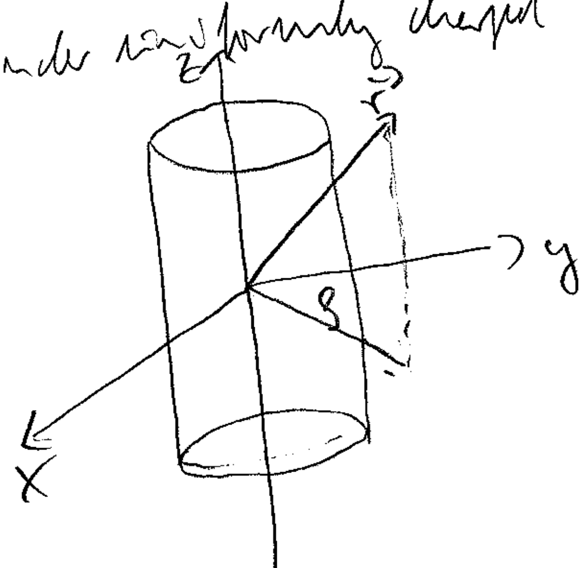
Problems Chapter V

(1) Cylinder uniformly charged



$n_q(\vec r) =$ const.

$n_q = \frac{\text{charge}}{\text{Volume}}$

Symmetries: consider yourself somewhere in space. Then you realize that you can't distinguish any difference whether you are at a point with $z = 0$ or any other value $z = a$. This means the electrical field can not depend on z. That's because the cylinder is infinitely long. Also there is no difference whether you go on a circle around the cylinder axis. Thus $|\vec E|$ can be only a function of $\rho = \sqrt{x^2+y^2}$

$|\vec E| = E(\rho)$. We can't say much about the direction. That we have to figure out from the fact that $\vec E$ has a potential, for which we also know $\boxed{V(\vec r) = V(\rho)}$. We use cylinder coordinates to investigate the problem further, because it is most appropriate for such cylinder symmetric situations.

Cylinder coordinates (Math helpdesk!)

$$\vec r(\rho, \varphi, z) = \rho(\cos\varphi\, \vec i_x + \sin\varphi\, \vec i_y) + z\, \vec i_z$$

The unit vectors:

$$\vec i_\rho = \frac{\partial \vec r}{\partial \rho}\frac{1}{g_\rho} \quad \text{with } g_\rho = \left|\frac{\partial \vec r}{\partial \rho}\right|$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\varphi \, \vec{i}_x + \sin\varphi \, \vec{i}_y$$

$$\Rightarrow g_\rho = 1 \quad \Rightarrow \quad \vec{i}_\rho = \frac{\partial \vec{r}}{\partial \rho} = \cos\varphi \, \vec{i}_x + \sin\varphi \, \vec{i}_y$$

$$\vec{i}_\varphi = \frac{1}{g_\varphi} \frac{\partial \vec{r}}{\partial \varphi}$$

$$\frac{\partial \vec{r}}{\partial \varphi} = \rho \left(-\sin\varphi \, \vec{i}_x + \cos\varphi \, \vec{i}_y \right) \Rightarrow g_\rho = \rho$$

$$\vec{i}_\varphi = \frac{1}{\rho} \frac{\partial \vec{r}}{\partial \varphi} = -\sin\varphi \, \vec{i}_x + \cos\varphi \, \vec{i}_y$$

$$\vec{i}_z = \frac{1}{g_z} \frac{\partial \vec{r}}{\partial z}$$

$$\frac{\partial \vec{r}}{\partial z} = \vec{i}_z \Rightarrow g_z = 1$$

We get:

$$\boxed{\begin{aligned} \vec{i}_r &= \cos\varphi \, \vec{i}_x + \sin\varphi \, \vec{i}_y \; ; \; g_r = 1 \\ \vec{i}_\varphi &= -\sin\varphi \, \vec{i}_x + \cos\varphi \, \vec{i}_y \; ; \; g_\varphi = r \\ \vec{i}_z &= \vec{i}_z \text{ (of course!)} \; ; \; g_z = 1 \end{aligned}}$$

Cylinder coordinates are orthogonal coordinates, because

$$\vec{i}_r \vec{i}_\varphi = \vec{i}_r \vec{i}_z = \vec{i}_\varphi \vec{i}_z = 0$$

$$\vec{i}_r \times \vec{i}_\varphi = \vec{i}_z \; ; \; \vec{i}_\varphi \times \vec{i}_z = \vec{i}_r \; ; \; \vec{i}_z \times \vec{i}_r = \vec{i}_\varphi$$

Relation between electric potential and $\vec{E}$

$$dV = -\vec{E}\, d\vec{r}$$
$$= -\vec{E}\left(\frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \varphi} d\varphi + \frac{\partial \vec{r}}{\partial z} dz\right)$$
$$= -\vec{E}\left(g_\rho \vec{i}_\rho d\rho + g_\varphi \vec{i}_\varphi d\varphi + g_z \vec{i}_z dz\right)$$

on the other hand

$$dV = \frac{\partial V}{\partial \rho} d\rho + \frac{\partial V}{\partial \varphi} d\varphi + \frac{\partial V}{\partial z} dz$$

and thus

$$E_r = \vec{E}\cdot\vec{i}_r = -\frac{1}{g_\rho}\frac{\partial V}{\partial \rho} = -\frac{\partial V}{\partial \rho}$$
$$E_\varphi = -\frac{1}{g_\varphi}\frac{\partial V}{\partial \varphi} = -\frac{1}{\rho}\frac{\partial V}{\partial \varphi}$$
$$E_z = -\frac{1}{g_z}\frac{\partial V}{\partial z} = -\frac{\partial V}{\partial z}$$

or

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial \rho}\vec{i}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\vec{i}_\varphi + \frac{\partial V}{\partial z}\vec{i}_z\right)}$$

We know 2 things about $\vec{E}$, namely that it has a potential (it's a conservative field)

$$\oint d\vec{r}\cdot\vec{E} = 0 \quad \text{for all closed paths}$$

and Gauss's law

$$\oint_{\partial V} d\vec{S}\cdot\vec{E} = \frac{Q_V}{\varepsilon_0}$$

(4)

Our symmetry arguments tell us that

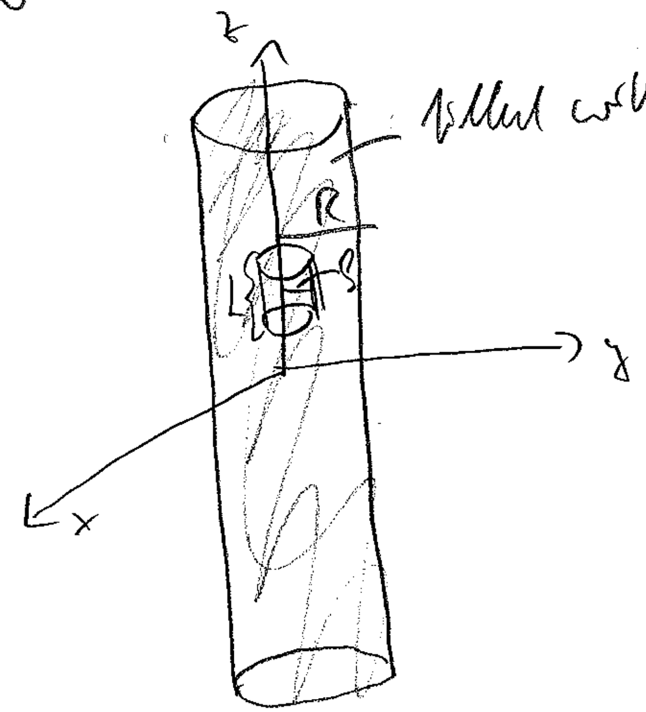
$$V = V(\rho) \quad (\underline{\text{not}} \text{ dependent on } \varphi \text{ or } z)$$

$$\Rightarrow \quad \vec{E} = -\frac{\partial V}{\partial \rho}\vec{i}_\rho$$

Note that $\vec{E}$ depends on φ but not on z, because only the magnitude must obey all the symmetries! The ansatz tells us that it's always pointing radially out of a pole from the cylinder. Thus finally let's us solve the problem!

$$\vec{E} = E_\rho \vec{i}_\rho$$

This we can easily plug into Gauss's law. We take a cylinder with radius ρ as volume. It's height is L:



For Gauß's law we need to find out, how much charge is in the little cylinder. Here we must be careful! If the Z_ρ is on side as shown in the figure, the amount of charge is:

$$Q_V = V_{Z_\rho} n_q = \pi \rho^2 L n_q \text{ if } \rho < R \quad \text{or} \quad Q_V = \pi R^2 L n_q \text{ if } \rho > R$$

For the electric flux we realise that we have to integrate over the top, the bottom and the mantle. (T, B, M).

Now for T and B, we have

$$d\vec{S}_T = -d\vec{S}_B \propto \vec{i}_z$$

$$\Rightarrow d\vec{S}_T \cdot \vec{E} = d\vec{S}_B \vec{E} = 0 \text{ because } \vec{i}_\rho \cdot \vec{i}_z = 0$$

For M we use the parametrization

$$M: \vec{r}(\varphi, z) = \rho \, \vec{i}_\rho(\varphi) + z \vec{i}_z$$

Then we get

$$d\vec{S} = \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial z} \, d\varphi \, dz = \rho \, \vec{i}_\varphi \times \vec{i}_z \, d\varphi \, dz = \rho \, \vec{i}_\rho \, d\varphi \, dz$$

$$\vec{E}(\vec{r}) = E(\rho) \, \vec{i}_\rho$$

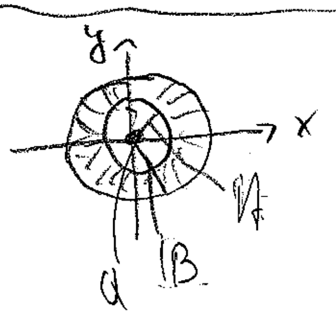
$$\Rightarrow \Phi^{(e)} = \int_M d\vec{S} \cdot \vec{E}(\vec{r}) = E(\rho)\rho \int_0^{2\pi} d\varphi \int_0^L dz = 2\pi\rho L E(\rho)$$

Gauß's law:

$$\frac{Q_V}{\varepsilon_0} = 2\pi \varrho L E(\varrho) = \frac{1}{\varepsilon_0}\begin{cases} \pi \varrho^2 L n_q & \text{for } \varrho < R \\ \pi R^2 L n_q & \text{for } \varrho > R \end{cases}$$

$$\Rightarrow \quad E(\varrho) = \frac{n_q}{2\varepsilon_0}\begin{cases} \varrho & \text{for } \varrho < R \\ \frac{R^2}{\varrho} & \text{for } \varrho > R \end{cases}$$

(2) Conducting sphere with hole in the middle



Slice of the situation in x y-plane

Since we want to have a static situation there must not be any $\vec{E}$-field in the conductor (since otherwise there would be a current density $\vec{j} = \sigma \vec{E}$, but that we'll learn only a little later.

Thus we have

$$\vec{E}(\vec{r}) = 0 \text{ for } |\vec{r}| \in (R_1, R_2)$$

As in the previous problem by symmetry (this time it's the full rotational symmetry or "isotropy") we know that the electric potential, V, can only depend on $r = |\vec{r}|$) and thus (remember spherical coordinates):

$$\vec{E} = -\frac{\partial V}{\partial r}\vec{e}_r \; ; \; V = V(r)$$

To apply Gauss's law we have to remember that in a conductor charge can flow around freely, and thus the charge Q (let's say it's positive) attracts the electrons in the conductor such that there is a surface charge of negative sign on the sphere of radius R_1 which must compensate the $\vec{E}$-field in the

conductor. Thus surface charge $\sigma_{\text{infl}}(\vec r)$ must be $=$ const along $|\vec r| = A$ (symmetry).
Now we can use Gauss's law for $|\vec r| < A$.
We put a sphere of radius $r < A$ as Gaussian volume. As in the lecture, we have:

$$\int_{\partial B_r} d\vec S \cdot \vec E = 4\pi r^2 E(r) = \frac{Q}{\varepsilon_0}$$

So we get

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r < A$$

Coulomb's law, as if there was no conductor.
Inside the conductor we have $\vec E = 0$, and thus for any B_r with $A < r < B$ Gauss's law tells us that

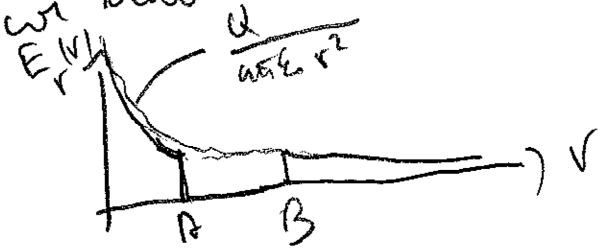
$$\int_{\partial B_r} d\vec S \cdot \vec E = \frac{Q_{B_r}}{\varepsilon_0} = 0$$

But $Q_{B_r} = Q + 4\pi A^2 \sigma_{\text{infl}} \Rightarrow \sigma_{\text{infl}} = -\frac{Q}{4\pi A^2}$

For $r > B$ we note that there must be an equal total positive charge Q on B_B (because of charge conservation).
Thus for B_r with $r > B$ we get by the same argument as above:

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{for } r > B$$

Thus we have



At A and B E_r jumps due to the influenced surface charge
@A: $E_r(A+0^+) - E(A-0^+) = \frac{\sigma_{\text{infl}}^{(A)}}{\varepsilon_0} = -\frac{Q}{4\pi\varepsilon_0 A^2}$

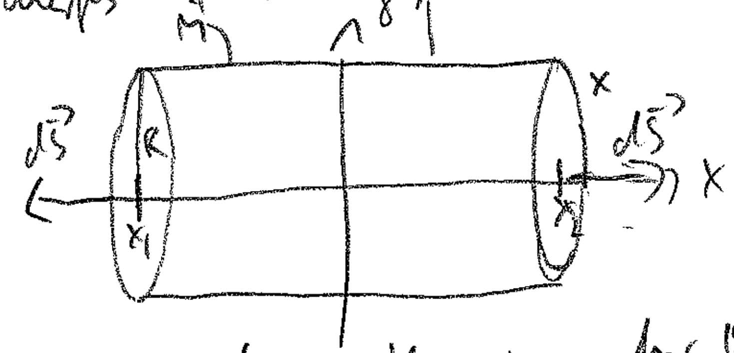
---

infl stands for influence which is the term for such charges as those which come from charges out such a conductor by motion of charges in the conductor to the surface and at B:

$$E_r(B+0^+) - E_r(B-0^+) = \frac{\sigma^{(B)}_{ind}}{\varepsilon_0} = + \frac{Q}{4\pi\varepsilon_0 B^2}$$

3. Doesn't work with our means. One must use the superposition principle and apply Gauss's law for each single charge and then add the results which leads back to Coulomb's law for the charges in question

4.



no flux through mantle, since for it $d\vec{S} \cdot \vec{E} = 0$, because $d\vec{S}\, \vec{e}_x = 0$.

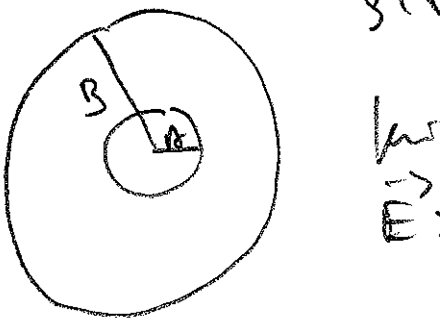
For the caps $\vec{E}$ is constant along them. Thus

$$\frac{Q_{cyl}}{\varepsilon_0} = \pi R^2 \alpha (x_2^2 - x_1^2)$$

$$Q_{cyl} = \pi R^2 \varepsilon_0 \alpha (x_2^2 - x_1^2)$$

5.

$$\rho(\vec{r}) = \begin{cases} c r & \text{for } A<r<B \\ 0 & \text{otherwise} \end{cases}$$



Gaussian volumes B_r

$\vec{E} = E_r(r)\, \vec{e}_r$

Inside: $r < A$

$E_r = 0$

$A < r < B$

$$Q_{B_r} = 4\pi \int_A^r dr\, r^2 c r = 4\pi c \int_A^r dr\, r^3 = \pi c (r^4 - A^4)$$

$$\Rightarrow \frac{\pi c(r^4 - A^4)}{\varepsilon_0} = 4\pi r^2 E(r)$$

$$\Rightarrow E(r) = \frac{c(r^4 - A^4)}{4\varepsilon_0 r^2}$$

<u>outside</u>

<u>$r > B$</u>

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{with } Q = \pi c (B^4 - A^4)$$

(6) gravitational field

$$\vec{G} = -\gamma \frac{m_1}{r^2} \vec{i}_r = -\gamma m_1 \frac{\vec{r}}{r}$$

where m_1 is a point mass in the origin. The proof goes as a chapter IV (or in the lecture), because the math is precisely the same. Thus

$$\int_{\partial V} d\vec{S}\, \vec{G} = -4\pi \gamma m_V = -4\pi\gamma \int_V dV\, \rho_m(\vec{r})$$

where m_V is the mass contained in V, and $\rho_m(\vec{r})$ is a general mass density.

Exercises Chapter 5

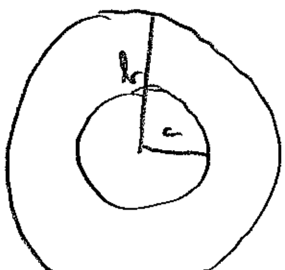
(1) The charge density is

$$n_Q(\vec{r}) = \frac{Q}{L^3 - l^3} = \text{const.}$$

Since Q is the total charge and $L^3 - l^3$ is the volume of the uniformly charged hollow cube. The volume of the cube with x on a side is x^3. The volume of the hole is taken out of the volume. Thus for $l < x < L$

$$Q_{cr} = n_Q (x^3 - l^3) = \frac{Q}{L^3 - l^3}(x^3 - l^3)$$

(2)



$$n_Q(\vec{r}) = \frac{Q}{\frac{4\pi}{3}(b^3 - a^3)}$$

Gaussian volume: Ball of radius r, B_r

Gauss's Law: $$\int_{\partial B_r} d\vec{S} \cdot \vec{E} = \frac{Q_{B_r}}{\varepsilon_0}$$

As applied in the lecture we use spherical coordinates and due to spherical symmetry

$$\vec{E}(\vec{r}) = E_r(r)\, \vec{i}_r$$

For $r < a$:

$$E_r(r)\, 4\pi r^2 = 0 \Rightarrow E_r(r) = 0$$

For $a < r < b$

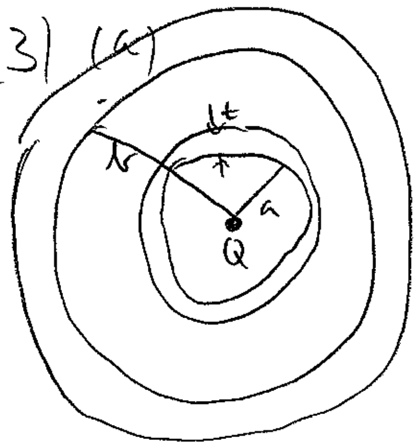
$$E_r(r)\, 4\pi r^2 = \frac{Q}{\cancel{\frac{4\pi}{3}}\, \varepsilon_0 (b^3 - a^3)} (r^3 - a^3) \cancel{\frac{4\pi}{3}} = \frac{Q}{\varepsilon_0} \frac{r^3 - a^3}{b^2 - a^3}$$

$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\,\frac{r^3-a^3}{b^3-a^3}$$

for $r > b$

$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

(3) (a)



Same as last exercise, but this time conductor-boundary conditions

$$E_r(r) = 0 \begin{cases} \text{if } a < r < a+t \\ \text{or } b < r < b+t \end{cases}$$

Gauss's Law gives

$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ for } 0 < r < a$$

and for $a+t < r < b$

and for $b+t < r$

(b) Q will spread uniformly over the sphere with radius $a+t$. At the surface at $r = b+t$ the charge $-2Q$ will be spread, but there is also an influence charge distribution with total charge $+Q$. There is also an influence charge on surface $r = b$ of total charge $-Q$.

With Gauss's law and the "conducting boundary conditions" one finds

$$E_r = 0 \text{ for } \begin{array}{l} 0 < r < a \\ a < r < a+t \\ b < r < b+t \end{array}$$

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ for } a+t < r < b$$

$$E_r = -\frac{Q}{4\pi\varepsilon_0 r^2} \text{ for } r > b+t$$

(4) As explained in Problem 1, in cylinder coordinates, we must have

$$\vec{E}(\vec{r}) = E_\rho(\rho)\, \vec{i}_\rho,$$

Conductor conditions for electrostatic situation:

$$E_\rho(\rho) = 0 \text{ for } a < \rho < b.$$ Gaussian volume cylinder with radius ρ.

In Gauss's law only the flux through the mantle is $\neq 0$ because the surface elements of the caps are $\perp$ to $\vec{E}$.

$$\Rightarrow \quad E_\rho(\rho) = \frac{\lambda}{2\pi\varepsilon_0 \rho} \text{ for } \rho < a \text{ and } \rho > b$$

(5) Since the sheet is a plane, there is symmetry w.r.t. translations in any plane $\parallel$ xy plane. Thus in Cartesian coordinates

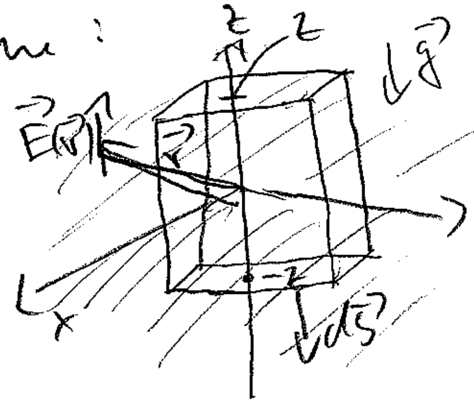
$$V(\vec{r}) = V(z)$$

$$\vec{E} = -\frac{\partial V}{\partial z}\vec{i}_z = E_z(z)\,\vec{i}_z \quad (1)$$

Since the situation is also symmetric w.r.t. reflections on the xy plane, we must have

$$E_z(z) = -E_z(-z)$$

Gaussian volume: cube of length $l = 2z$ with one surface $\parallel$ to xy plane:



$\sigma_Q(x,y) = \text{const}$ for $z=0$, 0 otherwise

σ_Q: charge per unit area on xy plane

Because of (1) only upper and lower surface of each contribute. Along those plane surfaces $\vec{E}$ is constant and thus:

$$\Phi^{(e)} = \vec{S}\cdot\vec{E} = (2z)^2\left[E_z(z) - E_z(-z)\right] \overset{\text{Gauss}}{=} \frac{4z^2}{\varepsilon_0}\sigma$$

Since $E_z(-z) = -E_z(z)$

$$\Rightarrow \quad 2\cdot 4z^2\cdot E_z(z) = \frac{4z^2}{\varepsilon_0}\sigma$$

$$\Rightarrow \quad \boxed{E_z(z) = \frac{\sigma}{2\varepsilon_0}}$$

The total force on charge Q is

$$\vec{F}_{el} + \vec{F}_G = 0$$

or $$Q E_z \vec{i}_z - mg\,\vec{i}_z = 0$$

or $$Q E_z = mg$$

Now $$Q E_z = \frac{\sigma Q}{2\varepsilon_0} = mg$$

$$\Rightarrow \quad \boxed{\sigma = \frac{2\varepsilon_0 mg}{Q}}$$