Renormalization and selfconsistency

The Φ-functional in relativistic quantum field theory

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Content

- Motivation
  - Thermodynamics of strongly interacting systems
  - Conservation laws, detailed balance, thermodynamical consistency
  - Finite width effects (resonance, damping, . . .)

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- Motivation
  - Thermodynamics of strongly interacting systems
  - Conservation laws, detailed balance, thermodynamical consistency
  - Finite width effects (resonance, damping, . . .)
- Concepts
  - Real time formalism
  - 2PI action
  - Equations of motion
  - Renormalization at finite temperature
  - Numerical solutions
  - Symmetries and trouble with 2PI formalism
- Summary and outlook

Real time formalism

- Initial statistical operator $\boldsymbol{\rho}_i$ at $t = t_i$
- Time evolution

$$\langle O(t)\rangle = \mathrm{Tr}\left[\boldsymbol{\rho}(t_i)\,\underbrace{\mathcal{T}_a\left\{\exp\left[+\mathsf{i}\int_{t_i}^{t}\mathrm{d}t'\,\mathbf{H}_I(t')\right]\right\}}_{\text{anti time–orderd}}\right.$$

$$\mathbf{O}_I(t)$$

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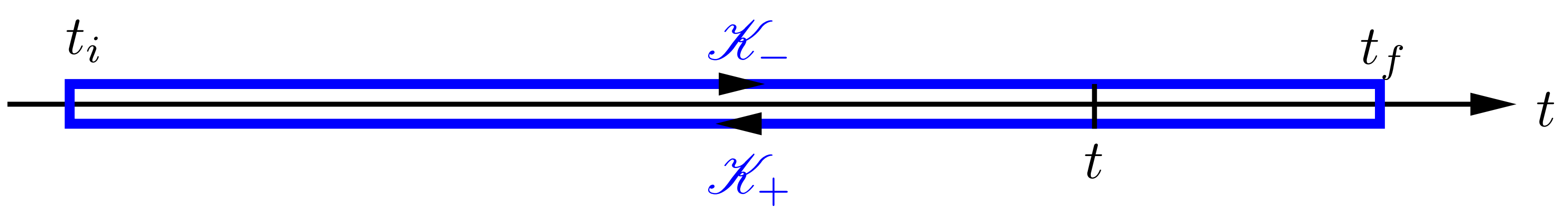
- Contour ordered Green's functions



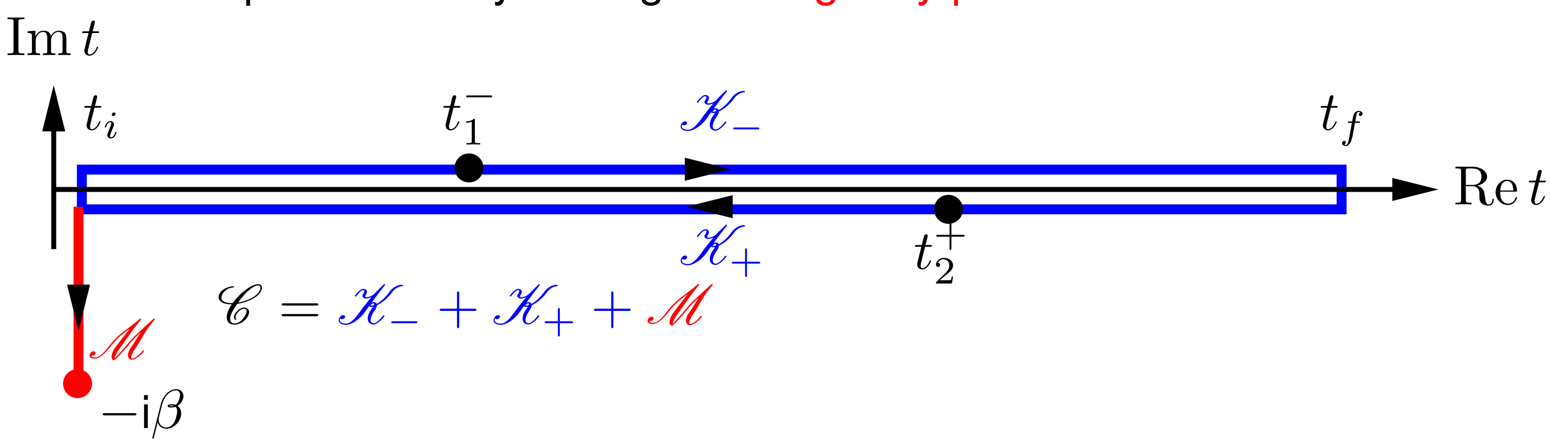
$$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+$$

Real-time formalism: Equilibrium

- In equilibrium

$$\boldsymbol{\rho} = \exp(-\beta \mathbf{H})/Z \text{ with } Z = \mathrm{Tr}\exp(-\beta \mathbf{H}), \quad \beta = 1/T$$

- Can be implemented by adding an imaginary part to the contour



- Correlation functions with real times: $\mathrm{i}G_{\mathscr{C}}(x_1^-, x_2^+)$
- Fields periodic (bosons) or anti-periodic (fermions) in imaginary time
- Feynman rules $\Rightarrow$ time integrals $\rightarrow$ contour integrals

2PI-formalism: The Φ-functional

- Introduce local and bilocal sources

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1 \phi_1\right\}_1 + \left\{\frac{\mathrm{i}}{2} K_{12} \phi_1 \phi_2\right\}_{12}\right]$$

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- Legendre transformation for φ and G:

$$\Gamma[\varphi, G] = W[J,K] - \{\varphi_1 J_1\}_1 - \frac{1}{2}\{(\varphi_1\varphi_2 + \mathrm{i} G_{12})K_{12}\}_{12}$$

2PI-formalism: The Φ-functional

- Saddle point expansion of path integral:

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- Equation of motion for the mean field φ and the “full” propagator G

$$-\Box\varphi - m^2\varphi := j = -\frac{\delta\Phi}{\delta\varphi}, \quad -\mathrm{i}(D_{12}^{-1} - G_{12}{}^{-1}) := -\mathrm{i}\Sigma = 2\frac{\delta\Phi}{\delta G_{21}}$$

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- Integral form of Dyson’s equation:

$$G_{12} = D_{12} + \{D_{11'} \Sigma_{1'2'} G_{2'2}\}_{1'2'}$$

- Closed set of equations of for φ and G

Diagrammar

- Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{m^2}{2}\vec{\phi}^2 - \frac{\lambda}{4!}(\vec{\phi}^2)^2$$

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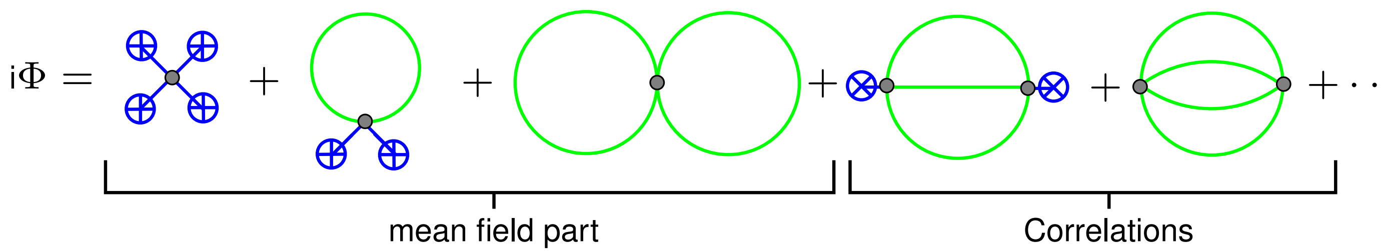

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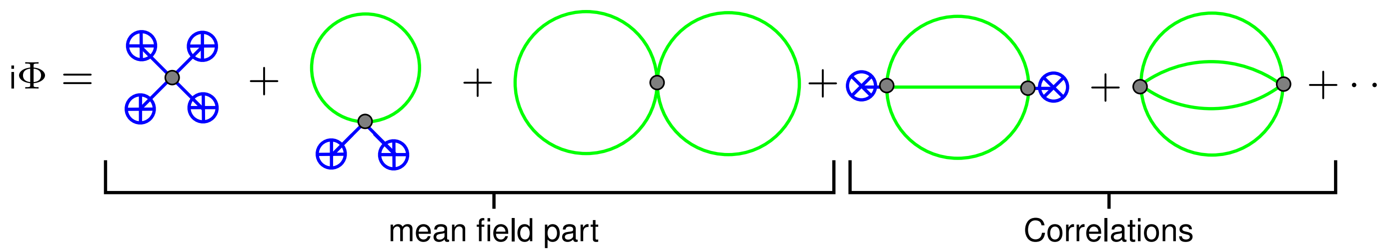
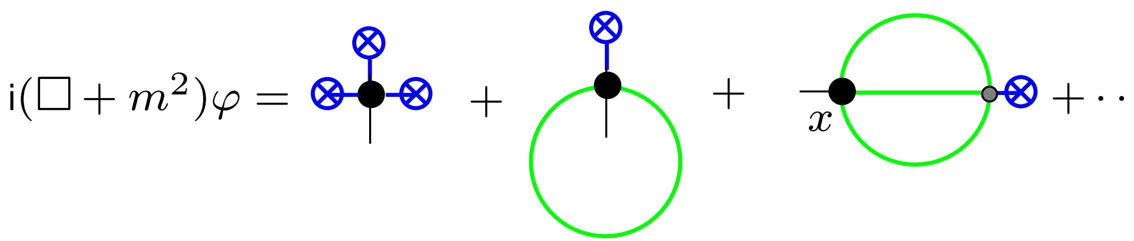


- Equation of motion for the mean fields



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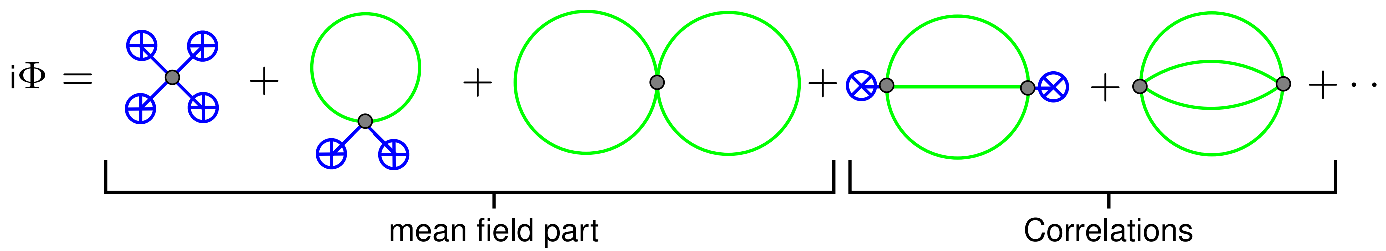
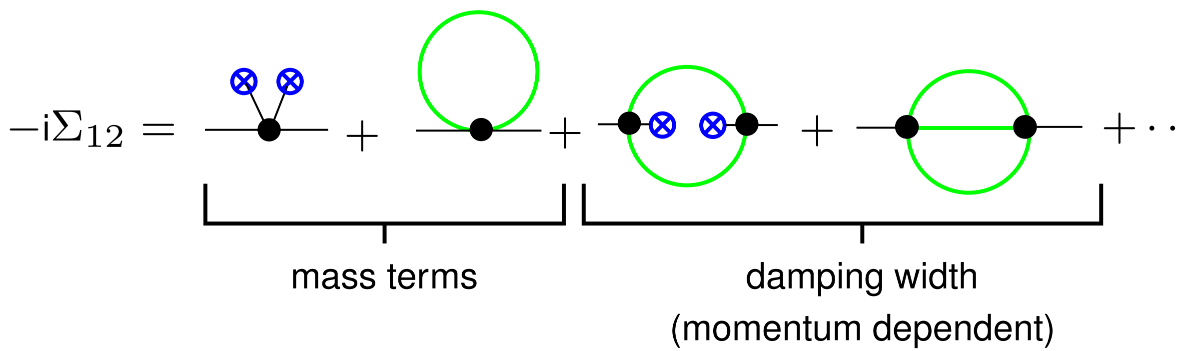


- Dyson equation for the Self-energy



2PI-formalism: Features

- Truncation of the Series of diagrams for Φ
- Expectation values for currents are conserved $\Rightarrow$ "Conserving Approximations"
- In equilibrium $\mathrm{i}\Gamma[\varphi, G] = \ln Z(\beta)$ (thermodynamical potential)
- consistent treatment of Dynamical quantities (real time formalism) and thermodynamical bulk properties (imaginary time formalism) like energy, pressure, entropy
- Real- and Imaginary-Time quantities "glued" together by Analytic properties from (anti-)periodicity conditions of the fields (KMS-condition)
- Self-consistent set of equations for self-energies and mean fields

How to renormalize and solve the equations of motion?

Why renormalization?

- Diagrams UV-divergent
- Control the physical parameters in vacuum: Masses, couplings
- “In-medium modifications” controlled from theory alone

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Difficulties compared to perturbation theory

- Self-consistency $\Rightarrow$ Resummation of infinitely many perturbative diagrams
- Diagrams do not show all divergences explicitely $\Rightarrow$ "hidden divergences"
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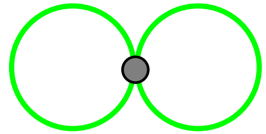
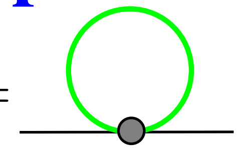
What about the numerics?

- Cannot use intermediate regularization which can be removed after renormalization
- BPHZ-Renormalization $\Rightarrow$ Get directly finite equations of motion
- But integrands have singularities

An example: Hartree approximation

$\Phi =$  $\Rightarrow \quad -\mathrm{i}\Sigma =$ 

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- “On-shell renormalization scheme”: m is mass of particles *in vacuo*

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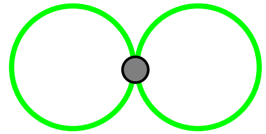
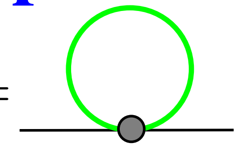
$\Phi =$  $\Rightarrow \quad -\mathrm{i}\Sigma =$ 

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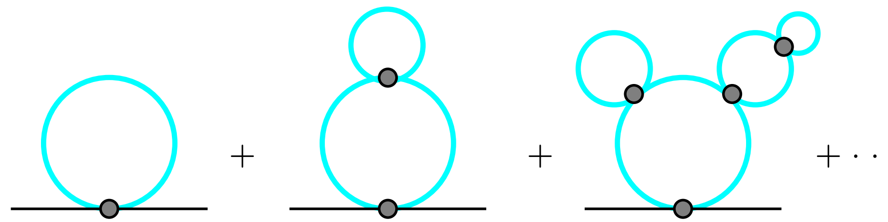

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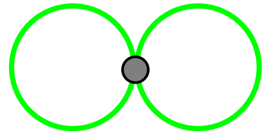
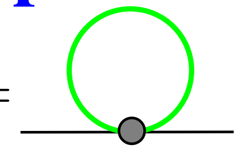
$\Phi =$  $\Rightarrow \quad -\mathrm{i}\Sigma =$ 

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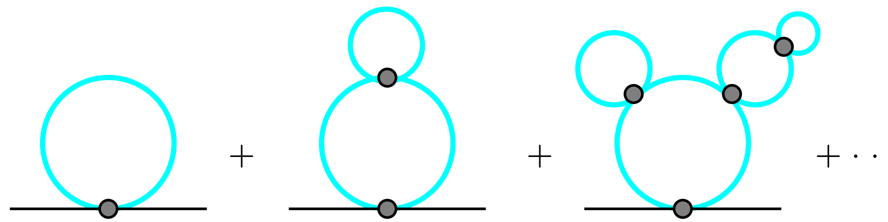
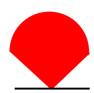
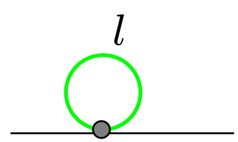
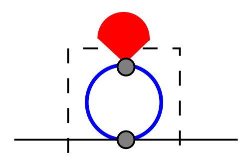
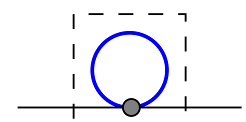


- Renormalized self-energy

$-\mathrm{i}\Sigma_{\text{ren}} =$  $=$  $-$  $-$ 

$\frac{\lambda}{2} G(l) \qquad \frac{\lambda}{2} G_v^2(l)\Sigma_{\text{ren}} \qquad \frac{\lambda}{2} G_v(l)$

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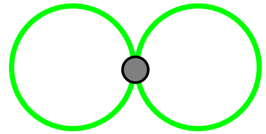
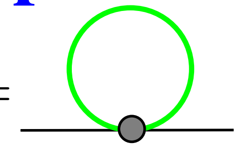
$\Phi =$  $\Rightarrow$ $-\mathrm{i}\Sigma =$ 

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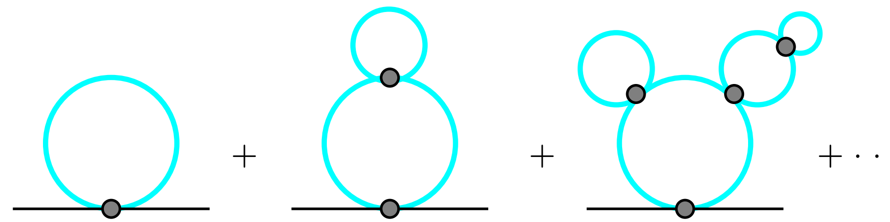

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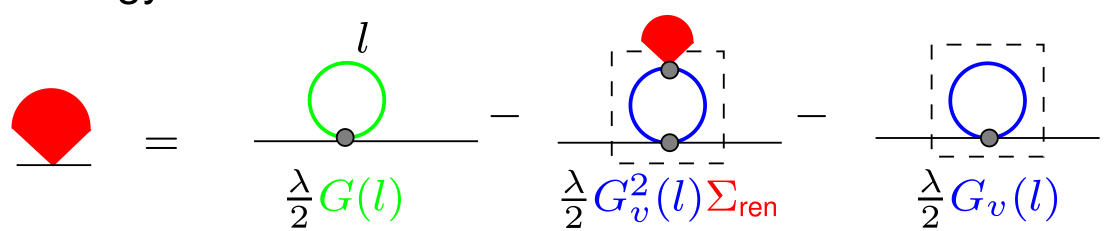
$-\mathrm{i}\Sigma_{\text{ren}} =$ 

- Result: Renormalized equation of motion, "gap equation":

$$M^2 = m^2 + \Sigma_{\text{ren}} = m^2 + \frac{\lambda}{32\pi^2}\left(M^2 \ln\frac{M^2}{m^2} - \Sigma_{\text{ren}}\right) + \underbrace{\frac{\lambda}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - M^2) n(p_0)}_{\to 0 \text{ for } T \to 0}$$

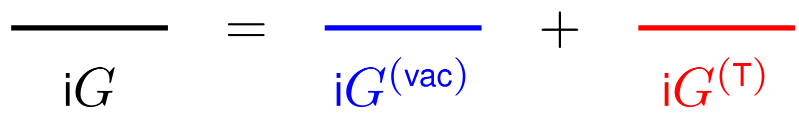
$n(p_0)$: Bose–Einstein distribution

Renormalization: General proof

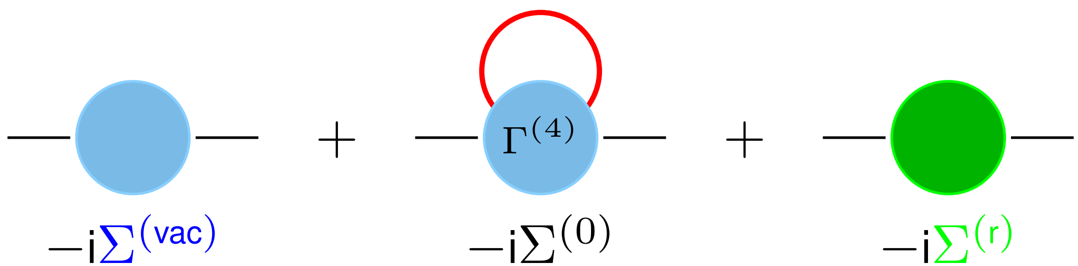
- Renormalization at $T = 0$
  - Power-counting for self-consistent propagators as in perturbation theory: $\delta = 4 - E$
  - Usual BPHZ-renormalization for wave function, mass and coupling constant
  - In practice: Use Lehmann-representation and dimensional regularization
  - Closed self-consistent finite Dyson-equations of motion
  - Numerically treatable

Renormalization: General proof

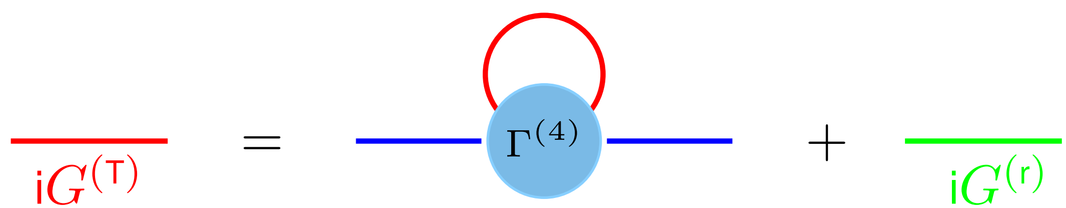
- Renormalization at finite temperature with vacuum counterterms
  - Split propagator in vacuum and T-dependent part



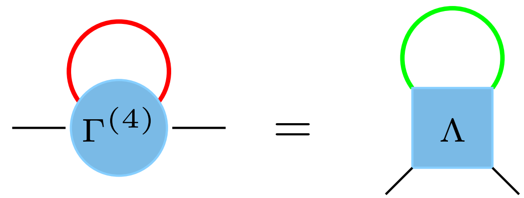
  - Expand self-energy around vacuum part

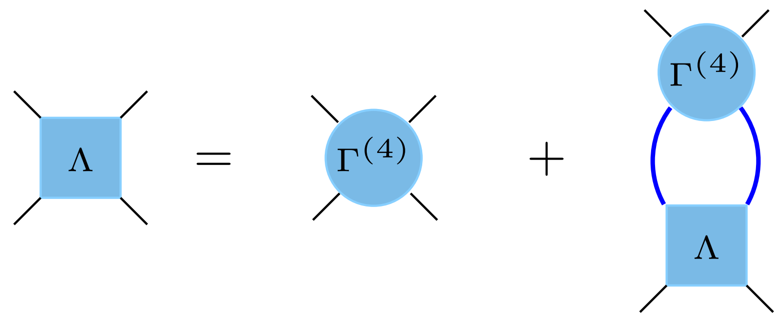


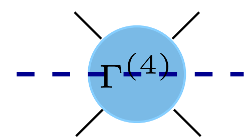
  - Need further splitting of propagator



Renormalization: General proof

- Renormalization of the four-point vertex
  - Σ^0 linear in $G^{(\mathrm{r})}$ $\Rightarrow$ 

  - Equation of motion $\Rightarrow$ 

  - s-channel Bethe-Salpeter equation: 

cuts more than three lines!
  - "BPHZ Boxes" in ladder-diagrams do not cut inside $\Gamma^{(4)}$.
  - Asymptotics + BPHZ-formalism: $\Gamma^{(4)}(l,p) - \Gamma^{(4)}(l,0) \cong O(l^{-\alpha})$ with $\alpha > 0$
  - Renormalized eq. of motion for Λ:

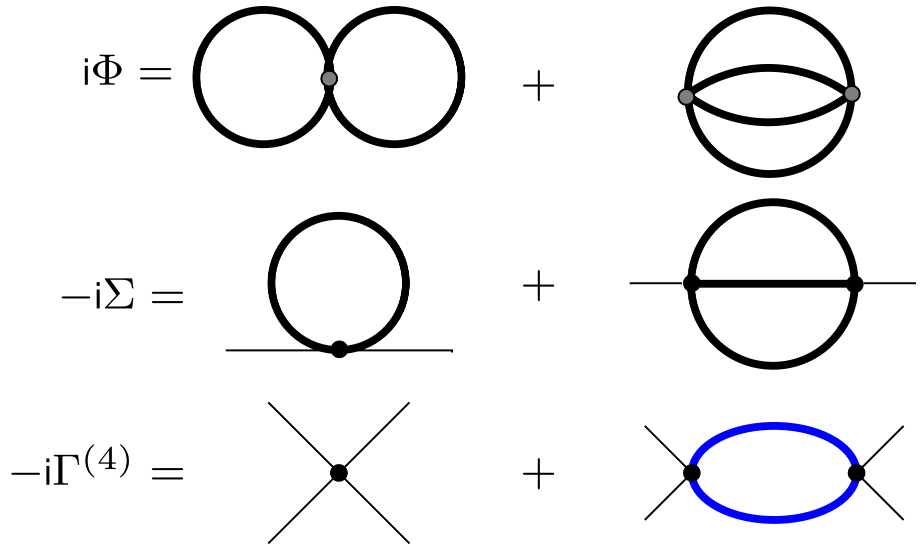
$$\Lambda(p,q) = \Lambda(0,0) + \Gamma^{(4)}(p,q) - \Gamma^{(4)}(0,0) + \mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}[\Gamma^{(4)}(p,l) - \Gamma^{(4)}(0,l)][G^{\mathrm{vac}}]^2(l)\Lambda(l,$$

$$+\,\mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\Lambda(0,l)[G^{\mathrm{vac}}]^2(l)[\Gamma^{(4)}(l,q) - \Gamma^{(4)}(l,0)]$$

  - Self-energy finite with vacuum counter terms

Example: tadpole and sunset

- The Φ-functional



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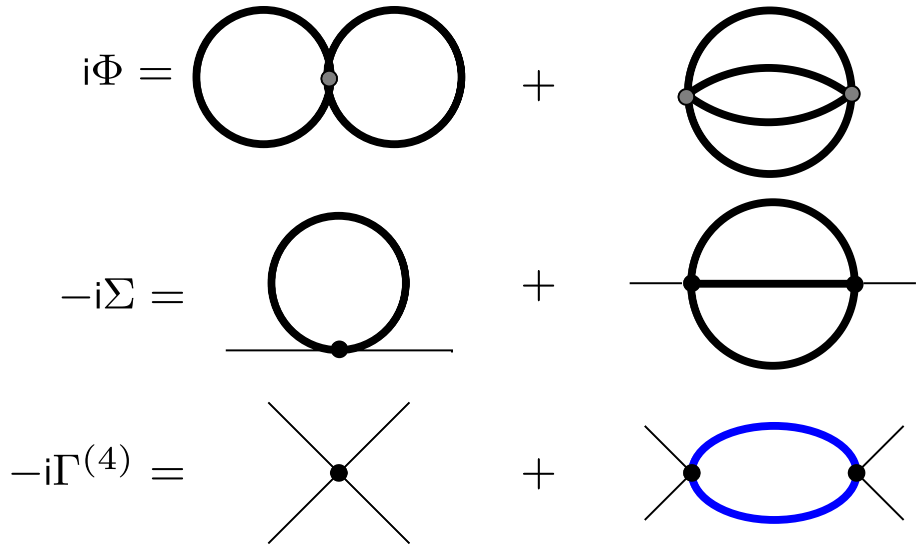
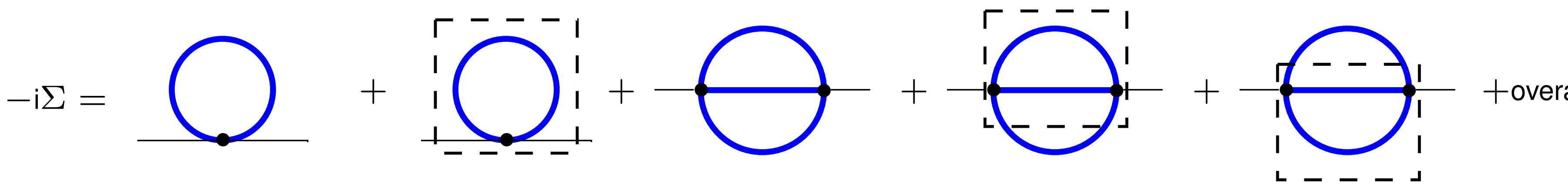


- The renormalized vacuum self-energy

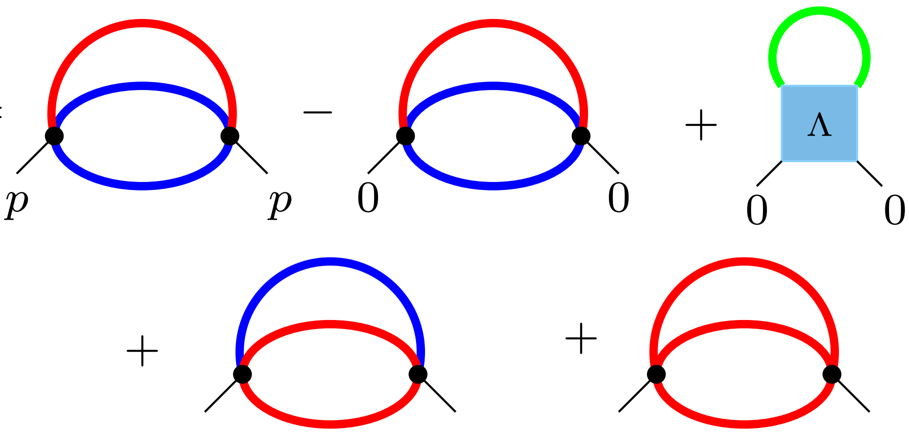


  - Numerics: Used dispersion (Lehmann) representation for propgators
  - $\Rightarrow$ renormalized kernels to be calculated by perturbative Feynman integrals
  - Renormalized equations of motion solved iteratively
  - Calculate $\Lambda(0, q)$ with the same techniques

Example: tadpole and sunset

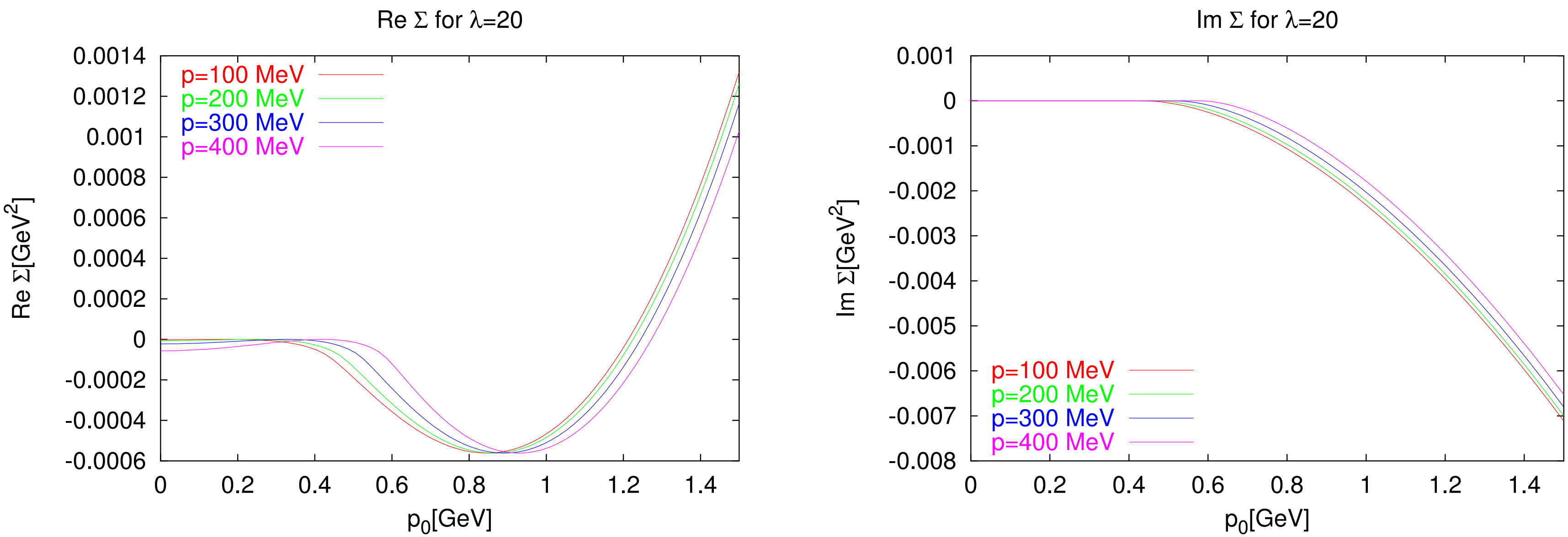
- Renormalization at finite temperature

$-\mathrm{i}\Sigma^{(\mathsf{T})}(p) =$



  - Only finite integrals
  - Numerics for three-dim integrals on a lattice in p_0 and $|\vec{p}|$
  - Equations of motion solved iteratively

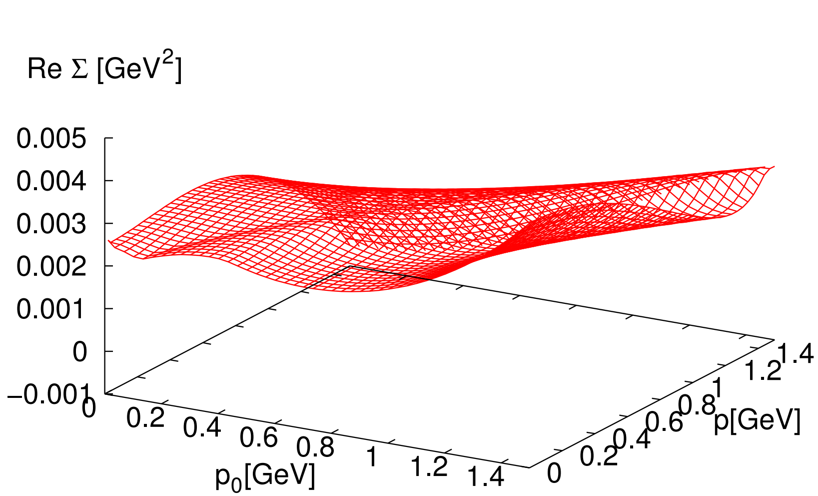
Results: the vacuum sunset self-energy



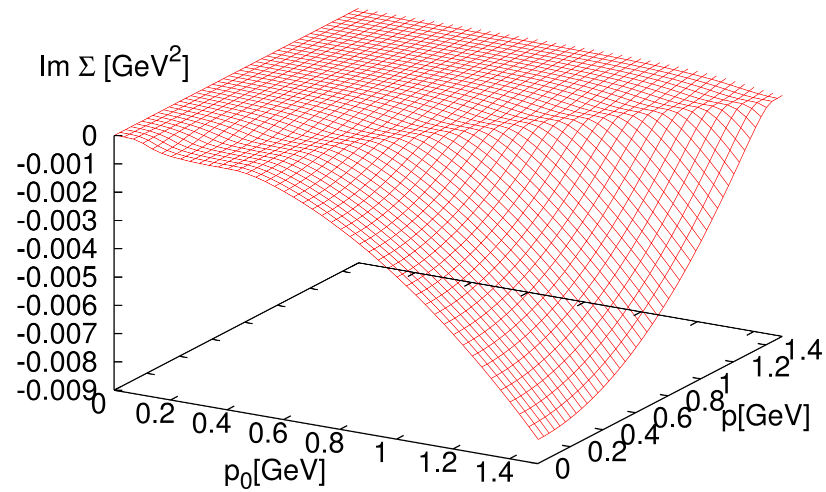
- Difference between perturbative and self-consistent calculation unvisible!
- Tadpole contribution “renormalized away” $\Rightarrow$ on-shell renormalization scheme
- Main contribution from the pole term of the propagator
- Threshold for continues part of the spectral function $\sqrt{s} = 3m$!

Results: sunset+tadpole diagrams at finite temperature

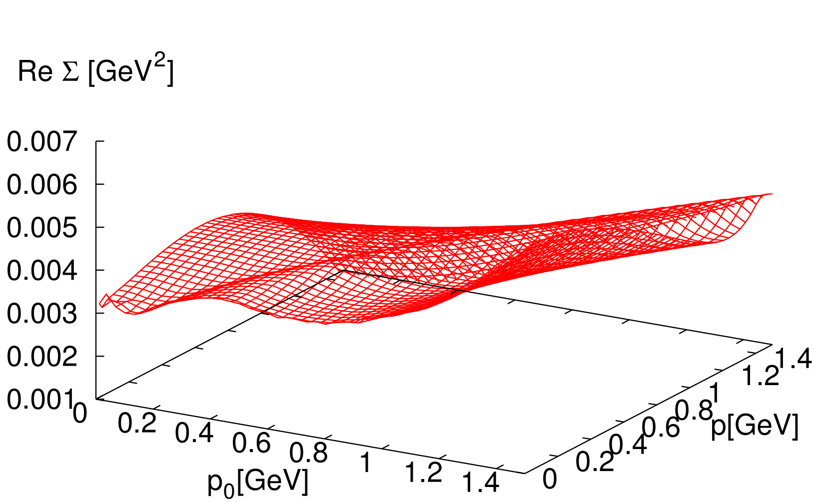
Pert. Re Σ for T=100MeV, λ=20



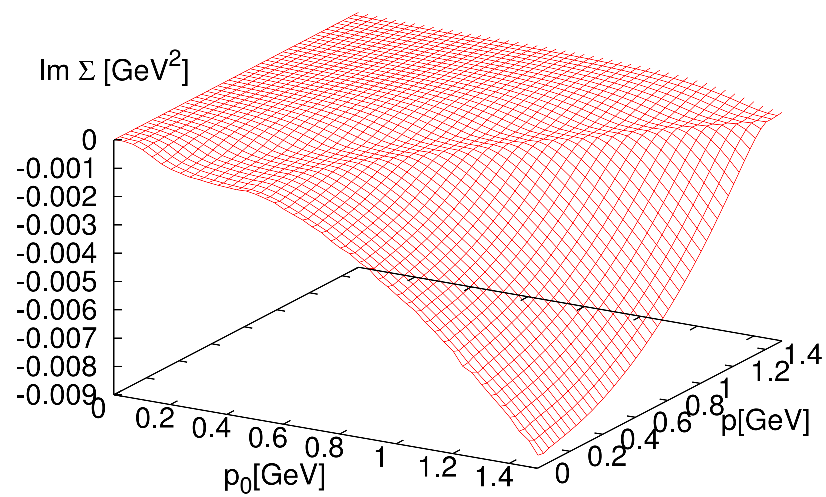
Pert. Im Σ for T=100MeV, λ=20



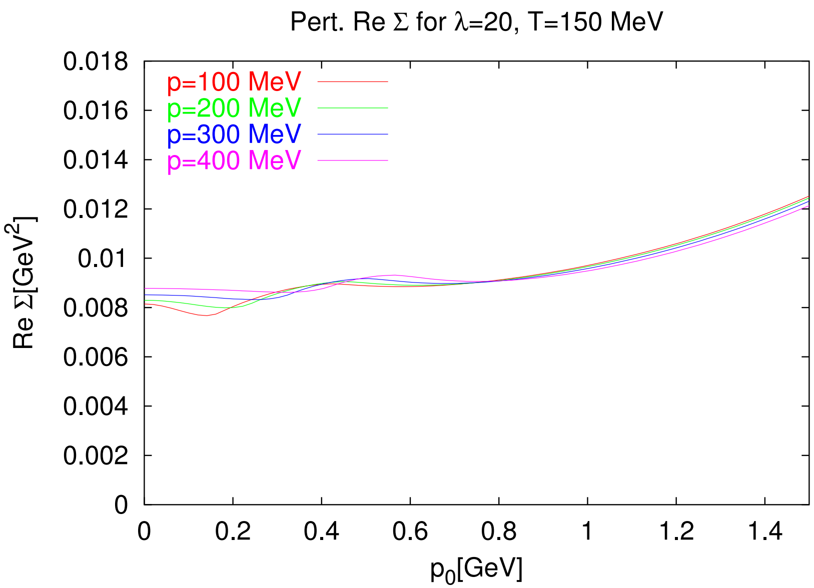
Re Σ for T=100MeV, λ=20

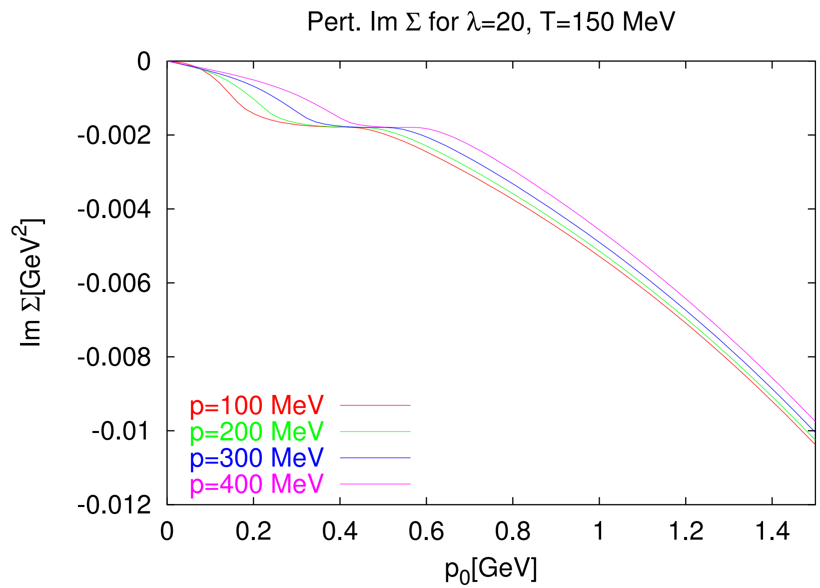


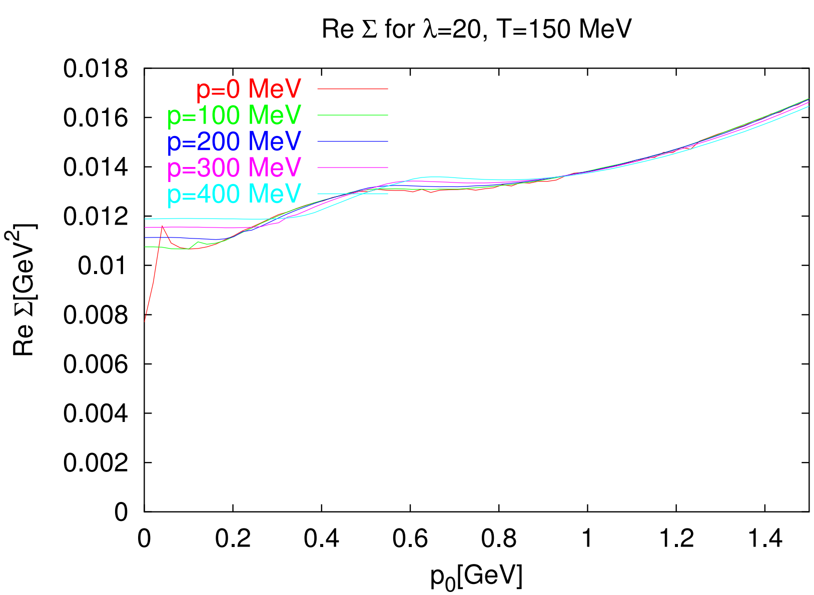
Im Σ for T=100MeV, λ=20

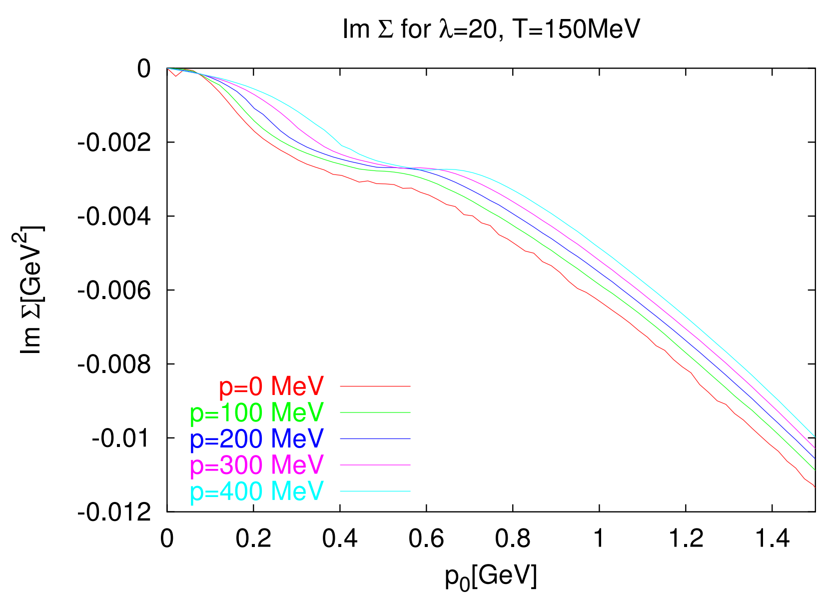


Results: sunset+tadpole diagrams at finite temperature

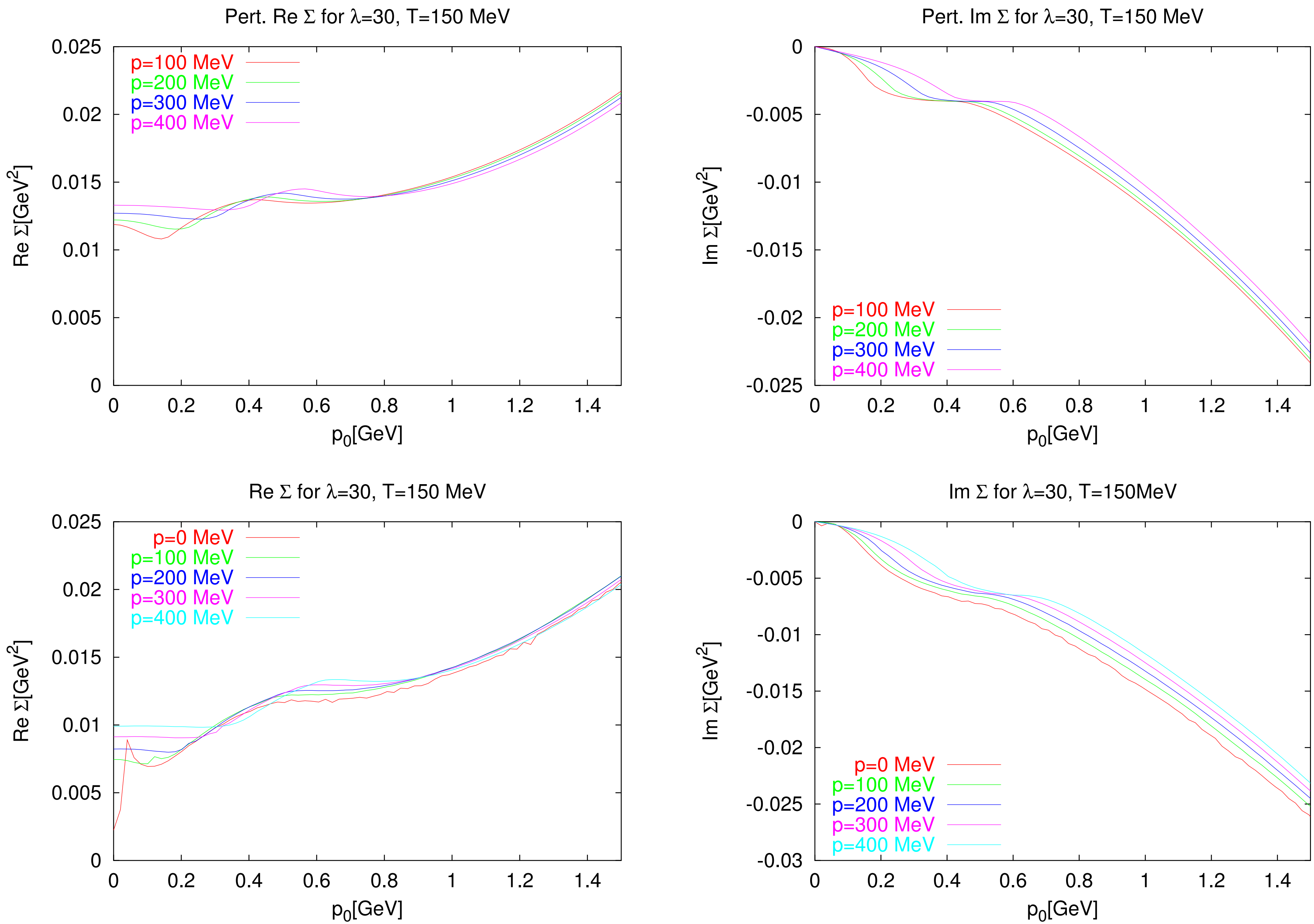








Results: sunset+tadpole diagrams at finite temperature



Effects of self-consistency

- Low-energy plateau in $\mathrm{Im}\,\Sigma$
- finite spectral width leads to a smoothing of "threshold" structures and a further increase in width
- counterbalanced by real part: tadpole term adds mass, which in the self-consistent treatment lowers the effective mass again
- for not too high couplings/temperature: sunset part adds spectral width which increases the self-consistent mass compared to the perturbative one
- for higher couplings/temperature: sunset contribution lowers the real part again compared to the perturbative result

Symmetry properties of Φ-derivable approximations

- Problem with Φ–Functional: Most approximations break symmetry!
- Reason: Only conserving for Expectation values for currents
- incomplete resummation leads to breaking of crossing symmetry
- Define Green's function at given mean field φ:

$$\left.\frac{\delta \mathbb{\Gamma}[\varphi, G]}{\delta G}\right|_{G=G_{\text{eff}}[\varphi]} \equiv 0$$

- Define new effective 1PI action functional

$$\Gamma_{\text{eff}}[\varphi] = \mathbb{\Gamma}[\varphi, G_{\text{eff}}[\varphi]]$$

Symmetry properties of Φ-derivable approximations

- Symmetry analysis $\Rightarrow \Gamma_{\text{eff}}[\varphi]$ symmetric functional in φ
- Stationary point

$$\left.\frac{\delta \Gamma_{\text{eff}}}{\delta \varphi}\right|_{\varphi=\varphi_0} = 0$$

- φ_0 and $G = G_{\text{eff}}[\varphi_0]$: self–consistent Φ–Functional solutions!
- Γ_{eff} generates external vertex functions fulfilling Ward–Takahashi identities
- External Propagator

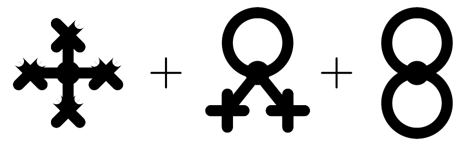
$$(G_{\text{ext}}^{-1})_{12} = \left.\frac{\delta^2 \Gamma_{\text{eff}}[\varphi]}{\delta \varphi_1 \delta \varphi_2}\right|_{\varphi=\varphi_0}$$

- G_{ext} generally not identical with Dyson resummed propagator

Example: Hartree approximation

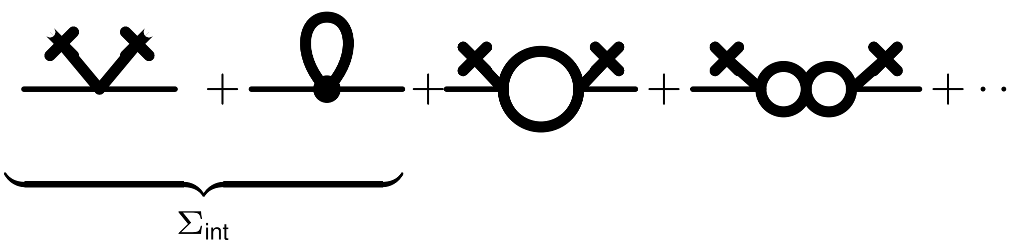
- Hatree approximation:

$i\Phi =$



- External self–energy defined on top of Hartree approximation

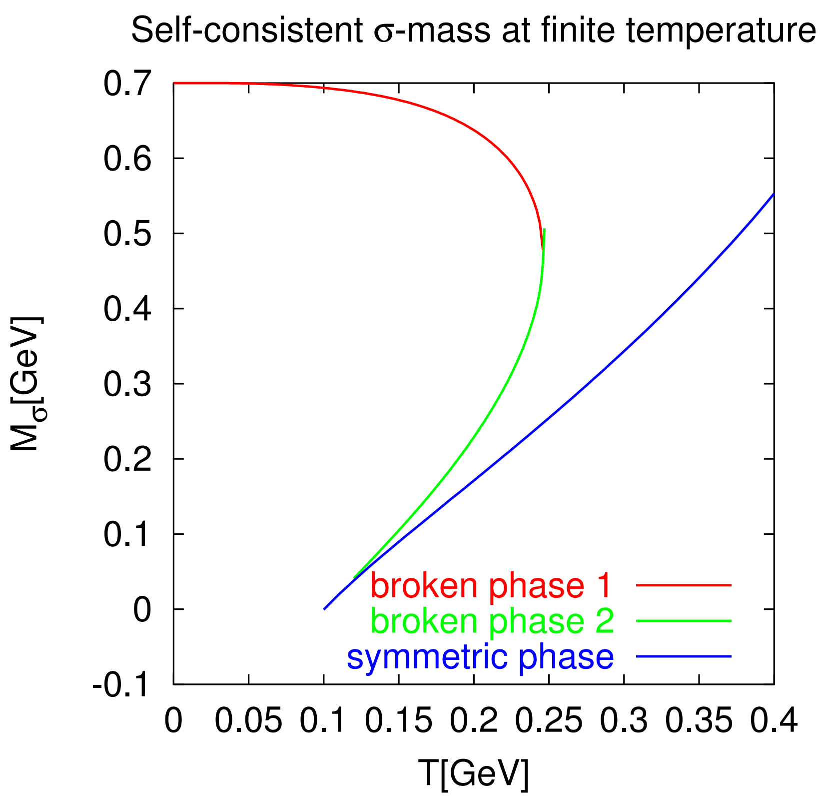
$-i\Sigma_{\text{ext}} =$

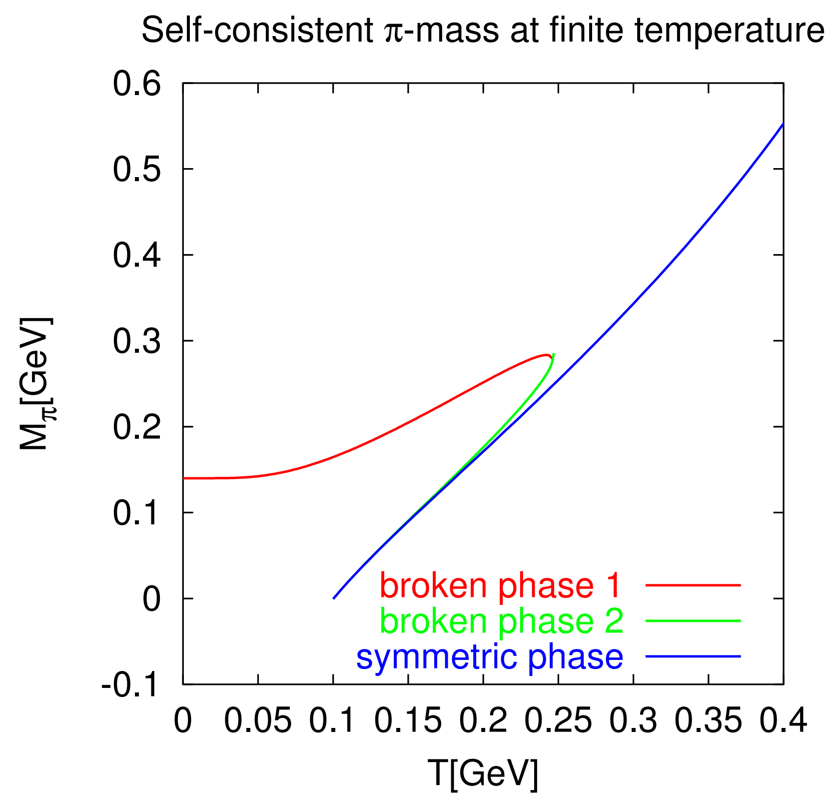


- Well-known result: RPA–Resummation restores symmetry
- Renormalization by the same counterterms as the self-consistent diagrams
- resums the crossing-symmetric channels missing in the self-consistent approximation
- in principle can be generalized to all Φ-derivable approximations

Numerical study of Hartree

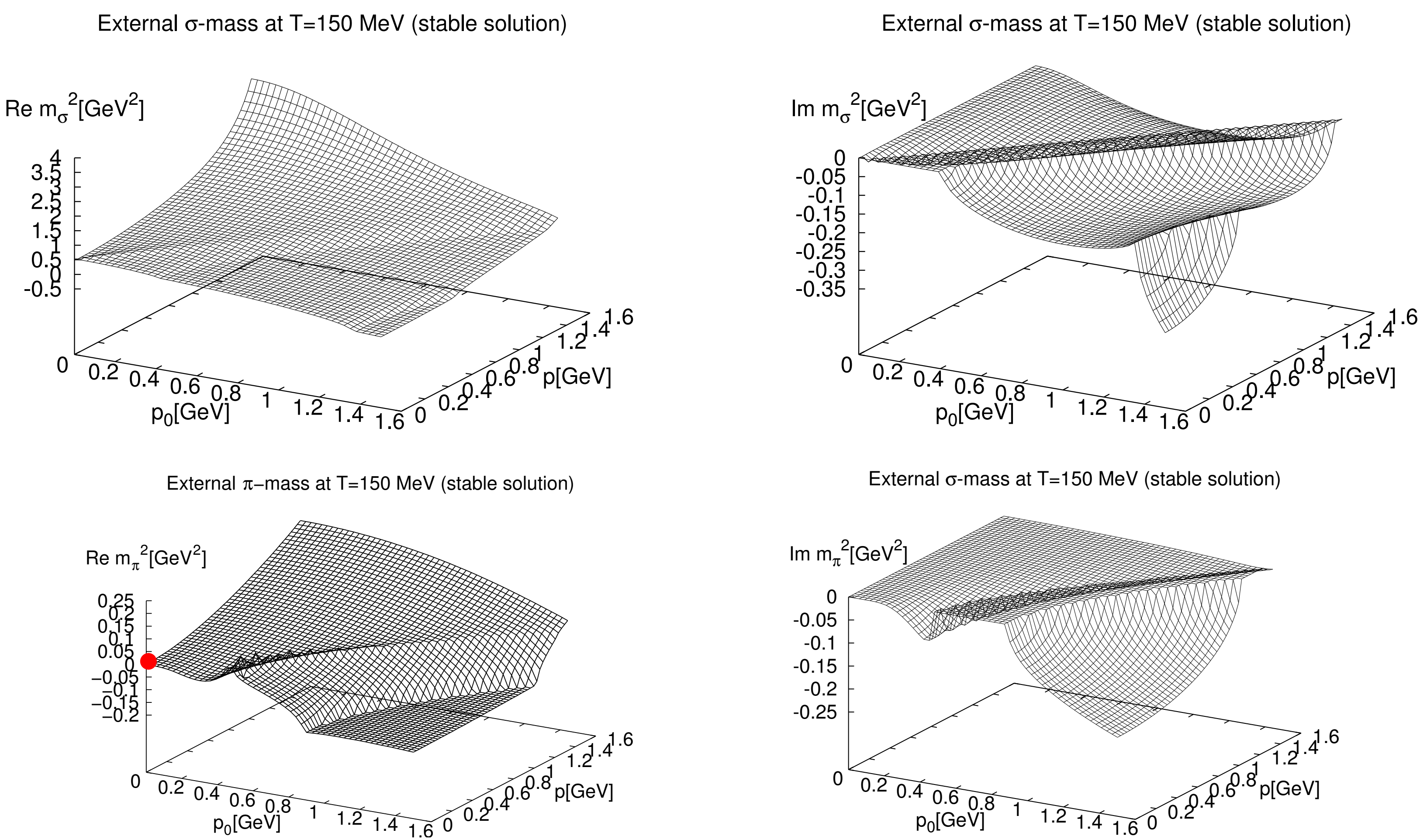
- Self-consistent masses for σ-meson (mode parallel to mean field) and the π-mesons (modes perpendicular to mean field)





- Ward-Takahashi-identity for self-energy $\Rightarrow$ Pions massless (Goldstone's theorem)
- Self-consistent approximation violates symmetries!

Numerical study of RPA-resummation



- Ward-Takahashi identity restored by RPA-resummation
- Internal lines of RPA-diagrams are the symmetry violating self-consistent propagators
- Remnants of symmetry violation: Wrong thresholds from non-zero masses of Goldstone modes

Conclusions and outlook

- Self-consistent Φ-derivable approximations: Renormalizable with temperature-independent counterterms
- Symmetry analysis and (partial) recovery of symmetries
- “Toolbox” for application to more realistic models
- Outstanding problem: Local gauge symmetries!
- First ideas: Projection to physical degrees of freedom
- For more details see `http://theory.gsi.de/~vanhees/index.html`