Self-consistent Conserving Approximations and Renormalization in Quantum Field Theory at Finite Temperature

Hendrik van Hees

in collaboration with Jörn Knoll

Contents

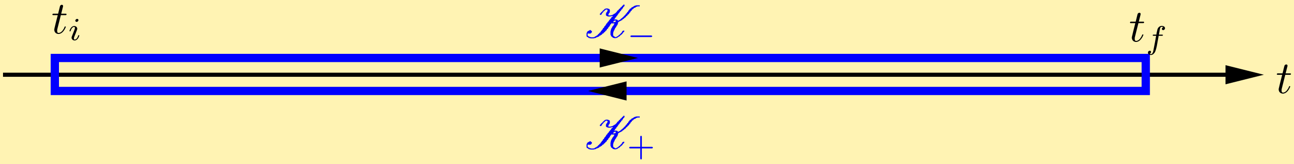
- Schwinger-Keldysh real-time formalism
- The Φ-derivable scheme
- Tadpole resummation for ϕ^4-theory
- The self-consistent sunset diagram
- Thermodynamical quantities
- Outlook

Schwinger-Keldysh Formalism

- Initial statistical operator $\boldsymbol{\rho}_i$ at $t = t_i$
- Time evolution of expectation values of observables:

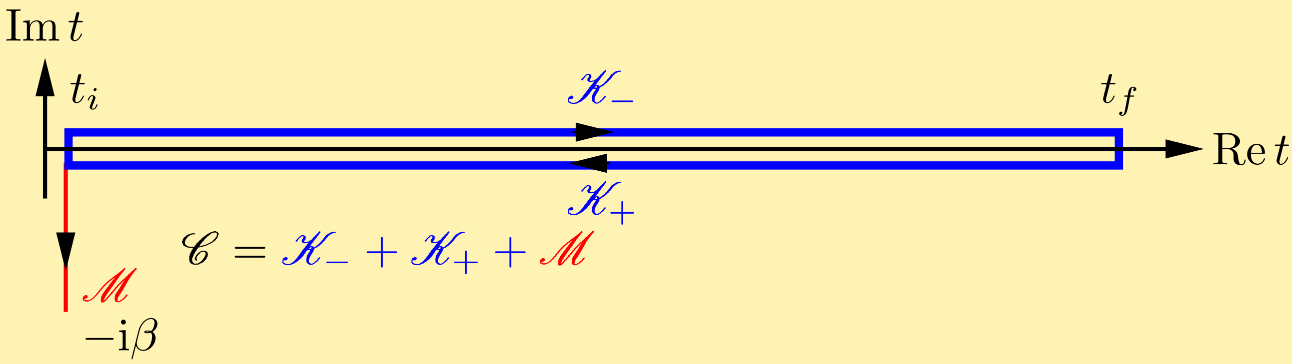
$$\langle O \rangle = \mathrm{Tr}[\boldsymbol{\rho}(t)\mathbf{O}(t)]$$

- Feynman rules
- Difference to vacuum: Contour-ordered Green's functions



$\mathscr{C} = \mathscr{K}_- + \mathscr{K}_+$

- In equilibrium: $\boldsymbol{\rho} = \exp(-\beta\mathbf{H})/Z$ with $Z = \mathrm{Tr}\exp(-\beta\mathbf{H})$
- Imaginary part of the time contour



- Fields periodic (bosons) or anti-periodic (fermions)

The Φ-Functional

- Introduce local and bilocal auxiliary sources
- Generating functional

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1\phi_1\right\}_1 + \left\{\frac{\mathrm{i}}{2}K_{12}\phi_1\phi_2\right\}_{12}\right]$$

- Generating functional for connected diagrams

$$Z[J,K] = \exp(\mathrm{i}W[J,K])$$

- The mean field and the connected Green's function

$$\varphi_1 = \frac{\delta W}{\delta J_1}, \ G_{12} = -\frac{\delta^2 W}{\delta J_1 \delta J_2} \Rightarrow \frac{\delta W}{\delta K_{12}} = \frac{1}{2}[\varphi_1\varphi_2 + \mathrm{i}G_{12}]$$

- Legendre transformation for φ and G:

$$\Gamma[\varphi,G] = W[J,K] - \left\{\varphi_1 J_1\right\}_1 - \frac{1}{2}\left\{(\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12}\right\}_{12}$$

- Exact saddle point expansion:

$$\begin{aligned}\Gamma[\varphi,G] =& S[\varphi] + \frac{\mathrm{i}}{2}\operatorname{Tr}\ln(-\mathrm{i}G^{-1}) + \frac{\mathrm{i}}{2}\left\{\mathscr{D}_{12}^{-1}(G_{12} - \mathscr{D}_{12})\right\}_{12} \\ & + \Phi[\varphi,G] \Leftarrow \text{all 2PI diagrams with at least 2 loops}\end{aligned}$$

$$\mathscr{D}_{12} = \left(-\Box - m^2 - \frac{\lambda}{2}\varphi^2\right)^{-1} \delta(x_1 - x_2)$$

Equations of Motion

- Physical solution defined by vanishing auxiliary sources:

$$\frac{\delta\Gamma}{\delta\varphi_1} = -J_1 - \{K_{12}\varphi_2\}_2 \stackrel{!}{=} 0$$

$$\frac{\delta\Gamma}{\delta G_{12}} = -\frac{\mathrm{i}}{2}K_{12} \stackrel{!}{=} 0$$

- Equation of motion for the mean field φ

$$-\Box\varphi - m^2\varphi - \frac{\lambda}{3!}\varphi^3 - \frac{\mathrm{i}}{2}\varphi\,\{G(x,x)\}_x + \frac{\delta\Phi}{\delta\varphi} = 0$$

- for the "full" propagator G $\Rightarrow$ Dyson's equation:

$$2\frac{\delta\Phi}{\delta G_{12}} = -\mathrm{i}(\mathscr{D}_{12}^{-1} - G_{12}{}^{-1}) := -\mathrm{i}\Sigma$$

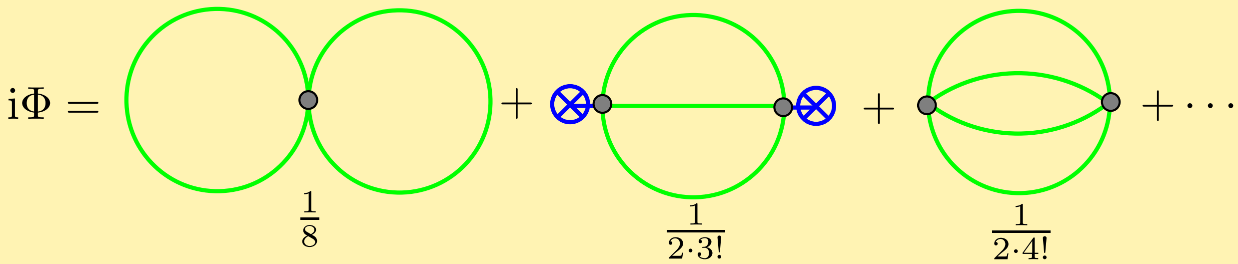
- Closed set of equations of for φ and G

"*Diagrammar*"

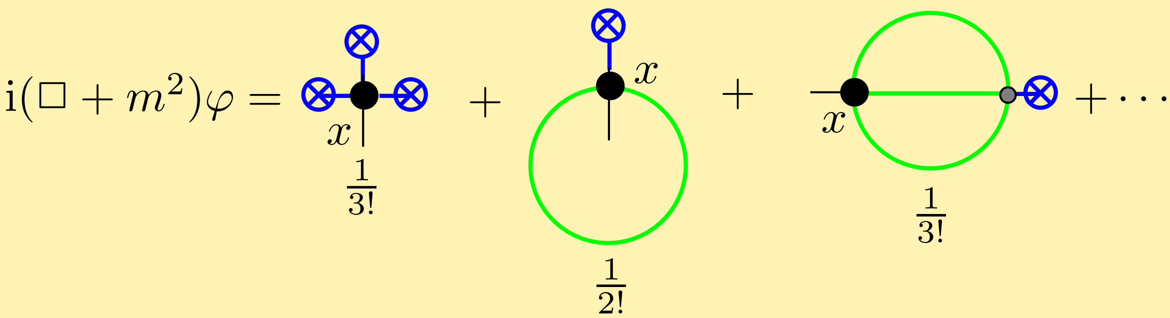
- ϕ^4-theory

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

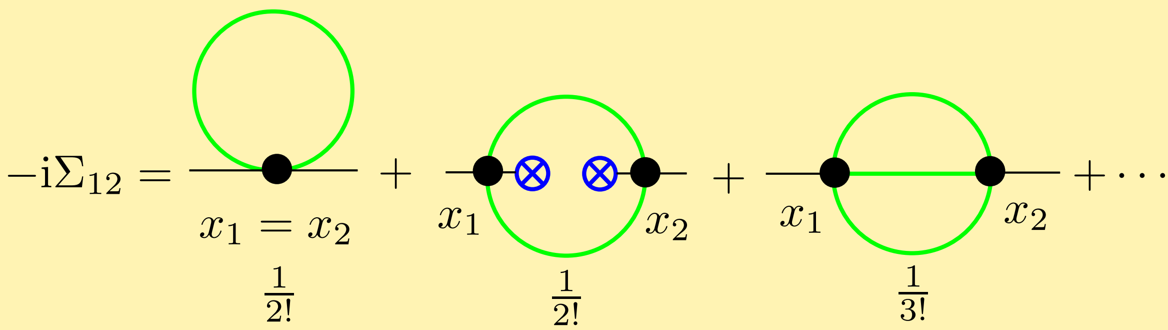
- 2PI Generating Functional



- Mean field equation of motion



- Self-energy



Properties of the Φ-Functional

Why using the Φ-functional?

- Series of diagrams for Φ truncated at a certain loop order

☞ Linearly realized Noether symmetries are respected

☞ Conserving Approximations (mod. anomalies)

- In equilibrium $\mathrm{i}\Gamma[\varphi, G] = \ln Z(\beta)$ (thermodynamical potential)
- consistent treatment of Dynamical quantities (real time formalism) and thermodynamical bulk properties (imaginary time formalism) like energy, pressure, entropy

- Real- and Imaginary-Time quantities "glued" together by Analytic properties from (anti-)periodicity conditions of the fields (KMS-condition)

- Self-consistent set of equations for self-energies and mean fields

Problem of Renormalization

Infinities and Renormalization

- UV-Divergences

☞ Need a renormalization technique for numerical solution of the self-consistent equations

- In terms of perturbation theory: Resummation of all self-energy insertions in propagators

☞ Self-consistent diagrams with explcit nested and overlapping sub-divergences

☞ Additional nested and overlapping sub-divergences from self-consistency

- Conjecture from Weinberg's theorem, BPHZ-renormalization

☞ At finite temperatures:
Self-consistent scheme rendered finite with local counterterms independent of temperature

- Analytic properties

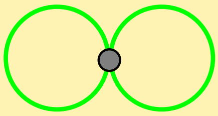
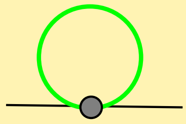
☞ subtracted dispersion relations

- Φ-Functional technique

☞ Consistency of counterterms

Self-Consistent Tadpole I

The tadpole approximation

$\Phi =$  $\qquad -\mathrm{i}\Sigma =$ 

- Here: Only time-ordered propagator needed
- The renormalized tadpole $d = 2\omega = 2(2-\epsilon)$:

$$-\mathrm{i}\Sigma = -\frac{\mathrm{i}\lambda}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\mu^{2\epsilon}\mathrm{i}G(p) + \mathrm{CT}$$

- Self-energy constant in p

☞ temperature dependent effective mass

- Dyson's equation can be resummed:

$$\mathrm{i}G(p) = \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\eta} + 2\pi n(p_0)\delta(p^2 - M^2)$$

$$\text{with } M^2 = m^2 + \Sigma,\ n(p_0) = \frac{1}{\exp(\beta|p_0|) - 1}$$

- Use standard formulae for dimensional regularized Feynman integrals:

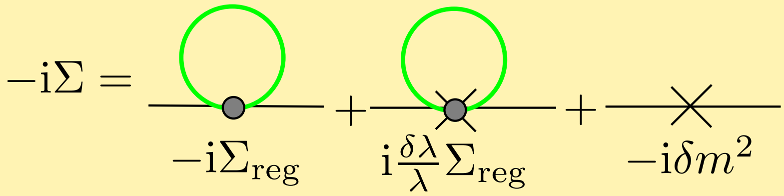
$$\Sigma_{\mathrm{inf}} = -\frac{\lambda}{32\pi^2}M^2\left[\frac{1}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{M^2}\right)\right] \tag{1}$$

- Does one need temperature dependent counter terms ($\propto \lambda M^2/\epsilon$)?

Self-Consistent Tadpole II

Counterterms

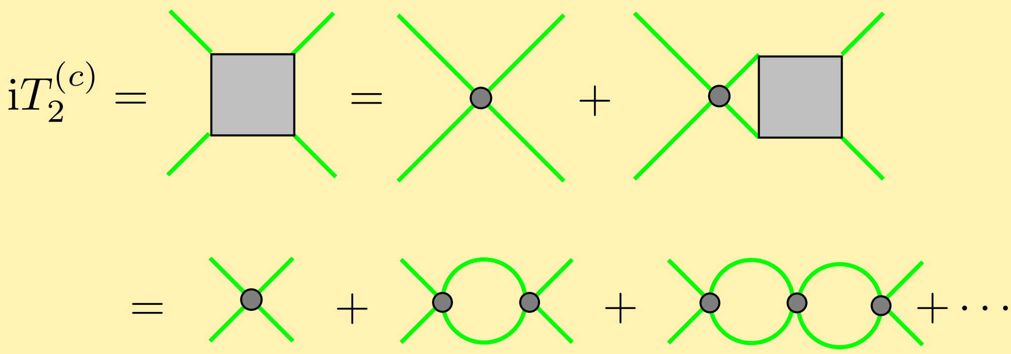
- Solution: We do not only need an overall but also a vertex counter-term for sub-divergence:



How to determine the vertex counterterm?

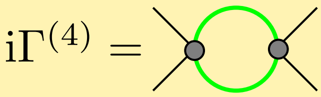
- From equations of motion

☞ consistency condition of “Bethe-Salpeter-type”:



- Renormalize the “Dinosaur diagram”

☞ vertex counterterm



The Finite Equation

The counterterms

- Renormalization conditions (physical scheme)

☞ $\Gamma^{(4)}_{\mathrm{vac}}(0) = 0$, $\Sigma_{\mathrm{vac}}(m^2) = 0$

- Counterterms:

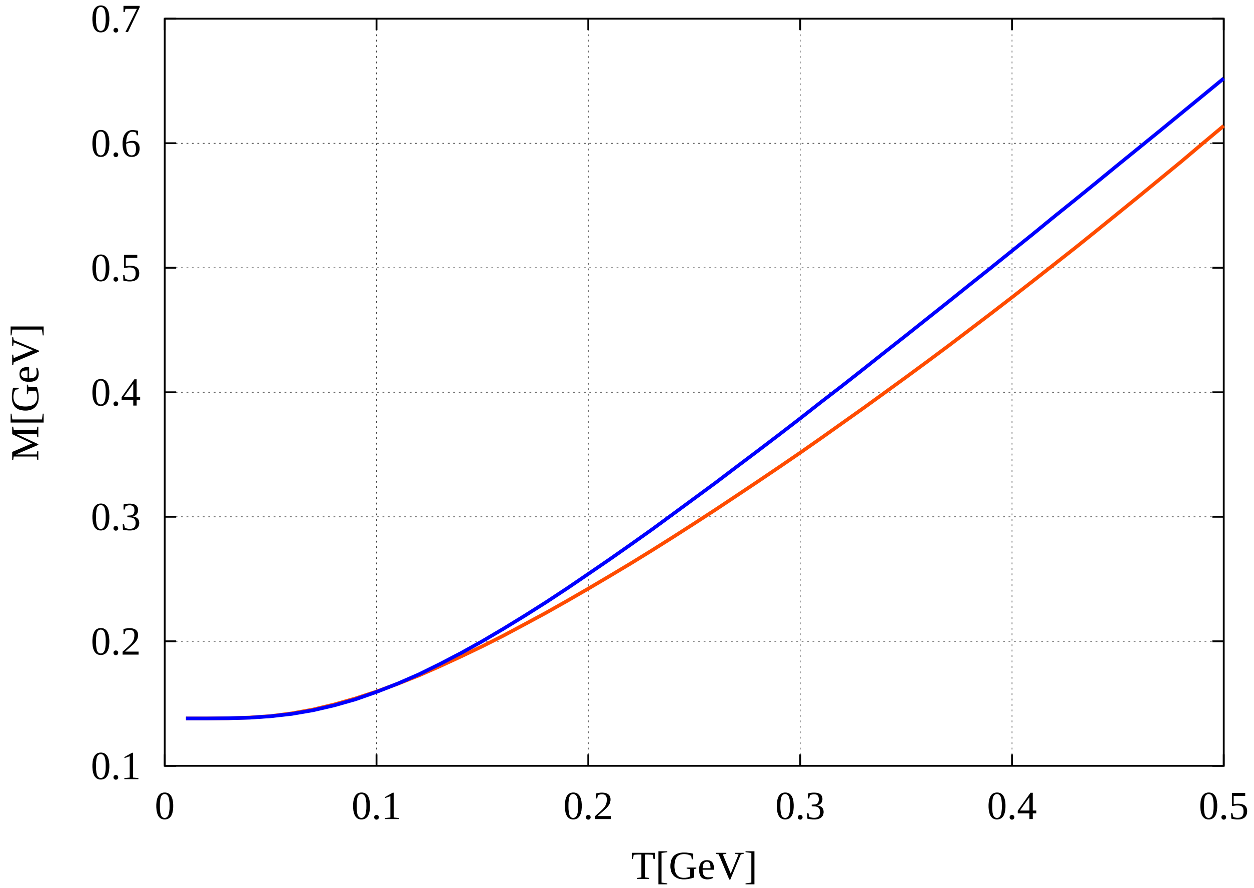
$$\delta m^2 = \frac{\lambda}{32\pi^2} m^2 \left[\frac{1}{\epsilon} - \gamma + 1 + \ln \frac{4\pi\Lambda^2}{m^2} \right]$$
$$\delta\lambda = -\frac{\lambda^2}{32\pi^2} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln \frac{4\pi\Lambda^2}{m^2} \right]$$

- Counterterms are independent of temperature and adjusted in vacuum

Self-consistent equation (gap equation)

$$M^2 = m^2 + \frac{\lambda}{32\pi^2} M^2 \ln \frac{M^2}{m^2} + \frac{\lambda}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - M^2) n(p_0)$$

Numerical Results

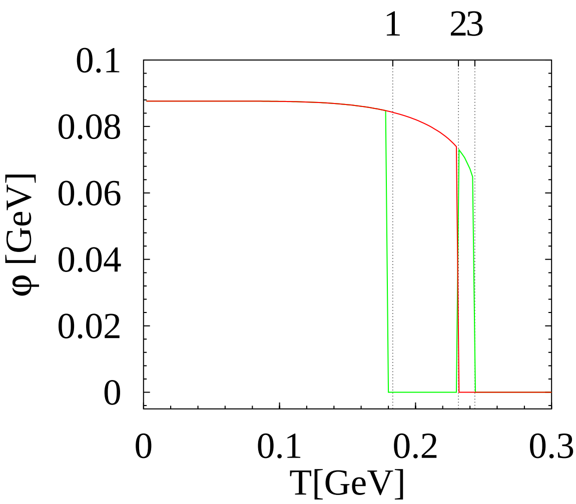


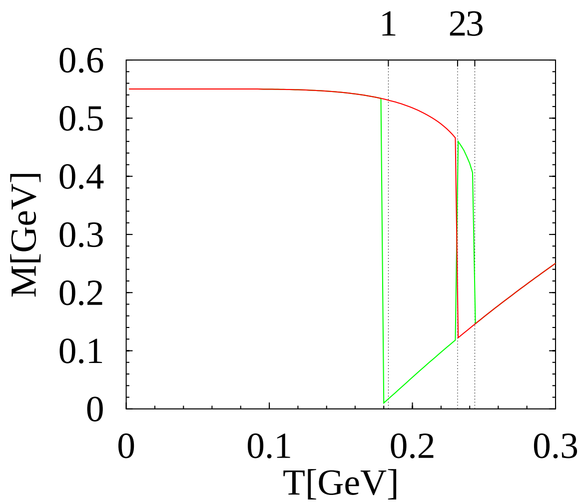
Numerical solution of the self-consistent tadpole equation compared to the perturbative result for $m = 140\text{MeV}$ and $\lambda = 50$

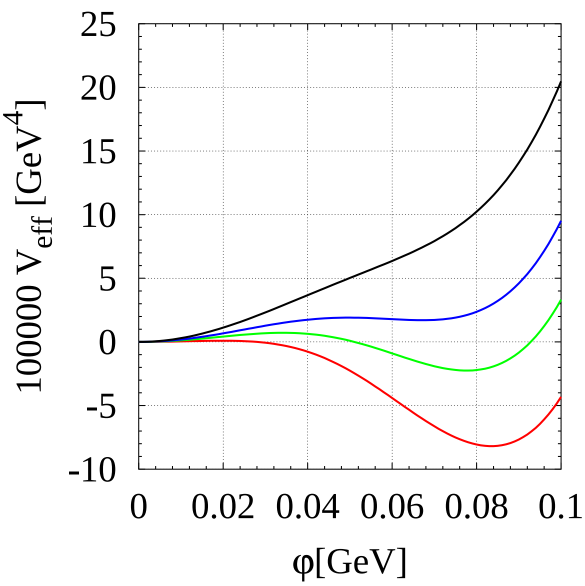
Spontaneously Broken Symmetry

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Stable (tree-level) vacuum at $\phi = \varphi = \frac{6\mu^2}{\lambda}$. Particles have $m^2 = +2\mu^2 > 0$
- Have to take the Mean Field φ into account and solve the coupled equations for both $\varphi = \text{const}$ and $\Sigma = \text{const}$. The renormalization procedure is the same as in the unbroken case.
- Parameters from linear the σ-model: $\mu = 400\text{MeV}$, $\lambda = 100$



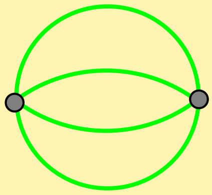
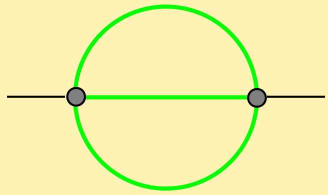




The effective potential for $T = 180\text{MeV}$ to $T = 280\text{MeV}$

The Sunset Diagram

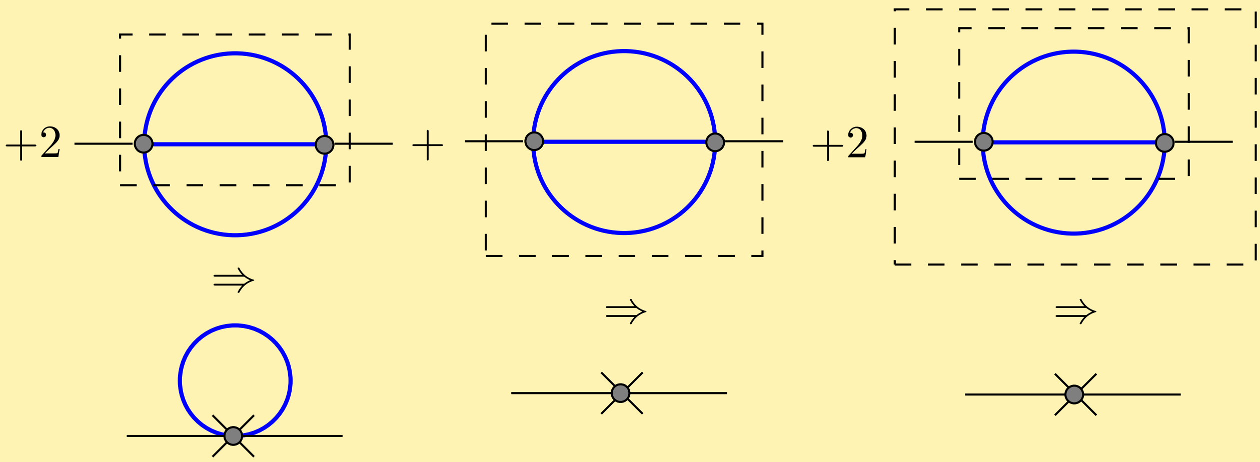
The vacuum part

$\mathrm{i}\Phi =$  $\Leftrightarrow$ $-\mathrm{i}\Sigma =$ 

$\frac{1}{2\cdot 4!}$ $\qquad$ $\frac{1}{3!}$

- Overall and sub-divergences to all orders perturbation theory

☞ Subtracted dispersion relations for vacuum divergences



- Counterterms of tadpole-type → const. and the overall subtraction → $O(p^2)$ (mass- and wave function renormalization)

☞ Simple subtractions in dispersion relation

Algorithm

Strategy for the vacuum

- Starting with perturbative propagators calculate imaginary part of the retarded self-energy (finite!):

$$\operatorname{Im}\Sigma_R = \frac{\Sigma^{-+} - \Sigma^{+-}}{2\mathrm{i}}$$

- Real part from twice subtracted dispersion relation:

$$\Sigma_R(s) = (s - m_{\text{ren}}^2)^2 \int_0^\infty \frac{\mathrm{d}z}{\pi} \frac{\operatorname{Im}\Sigma_R(z)}{(z - s + \mathrm{i}\epsilon\sigma(p_0))(z - m_{\text{ren}}^2)^2} +$$

$$+ \Sigma_R'(m_{\text{ren}}^2)(s - m_{\text{ren}}^2) + \Sigma_R(m_{\text{ren}}^2)$$

- Dyson's equation for the retarded propagator

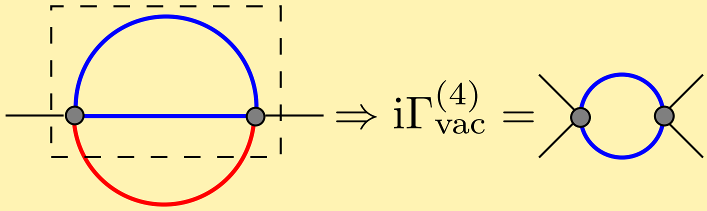
$$G_R = \frac{1}{p^2 - m^2 - \Sigma_R + \mathrm{i}\epsilon\sigma(p_0)}$$

  and plug it into the next calculation for $\operatorname{Im}\Sigma_R$.

- Iterate this procedure until the Σ_R does not change anymore

The Sunset Diagram for $T > 0$

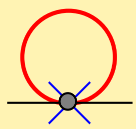
Renormalization of vacuum sub-divergence



Algorithm

- Calculate renormalized $\Gamma^{(4)}_{\text{vac}}$ with already given self-consistent vacuum propagator (blue)
- Renormalization condition: $\Gamma^{(4)}_{\text{vac}}(s=0)=0$.
- Calculate $\operatorname{Im}\Sigma_R$ with the full thermal propagator
- Subtract the vacuum part and $3\times$ the "bad diagram"
- Dispersion relation without subtraction for the rest

☞ Only pure vacuum subtractions in this part



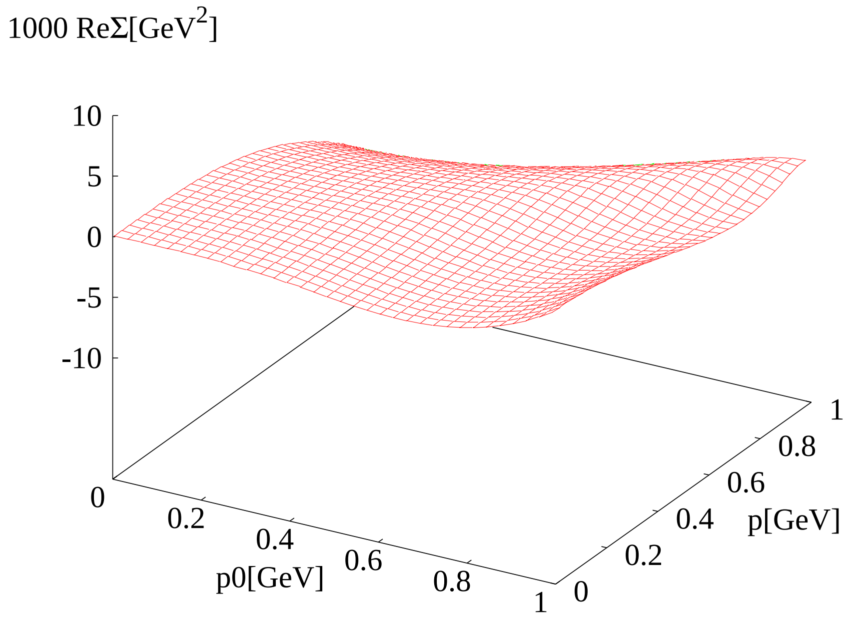
- Full retarded "bad diagram" with vacuum subtracted $\Gamma^{(4)}$-insertion

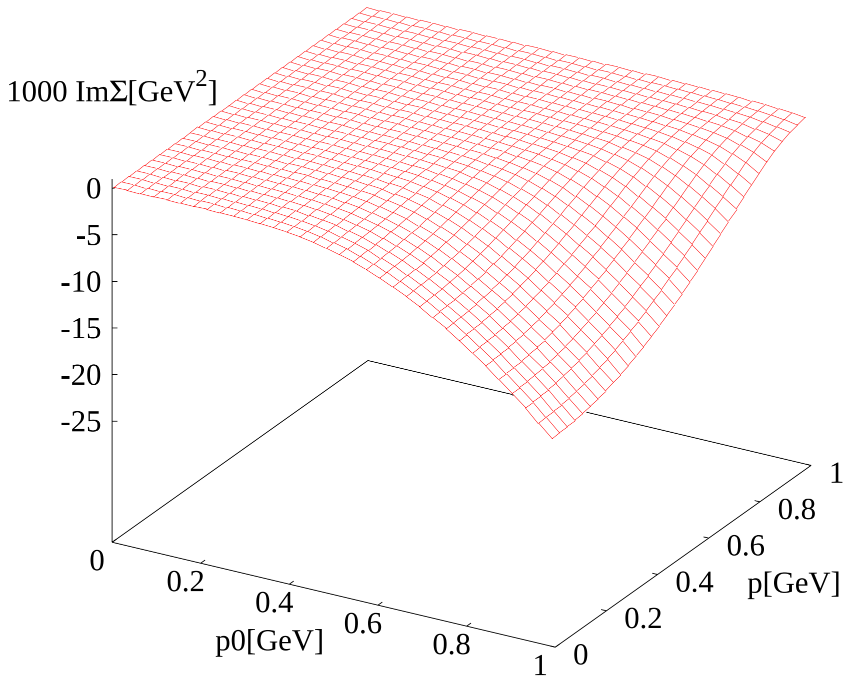
☞ finite

☞ Only vacuum sub-divergence subtracted

- Result: explicit overlapping divergences of the sunset diagram are completely subtracted with pure vacuum counter terms

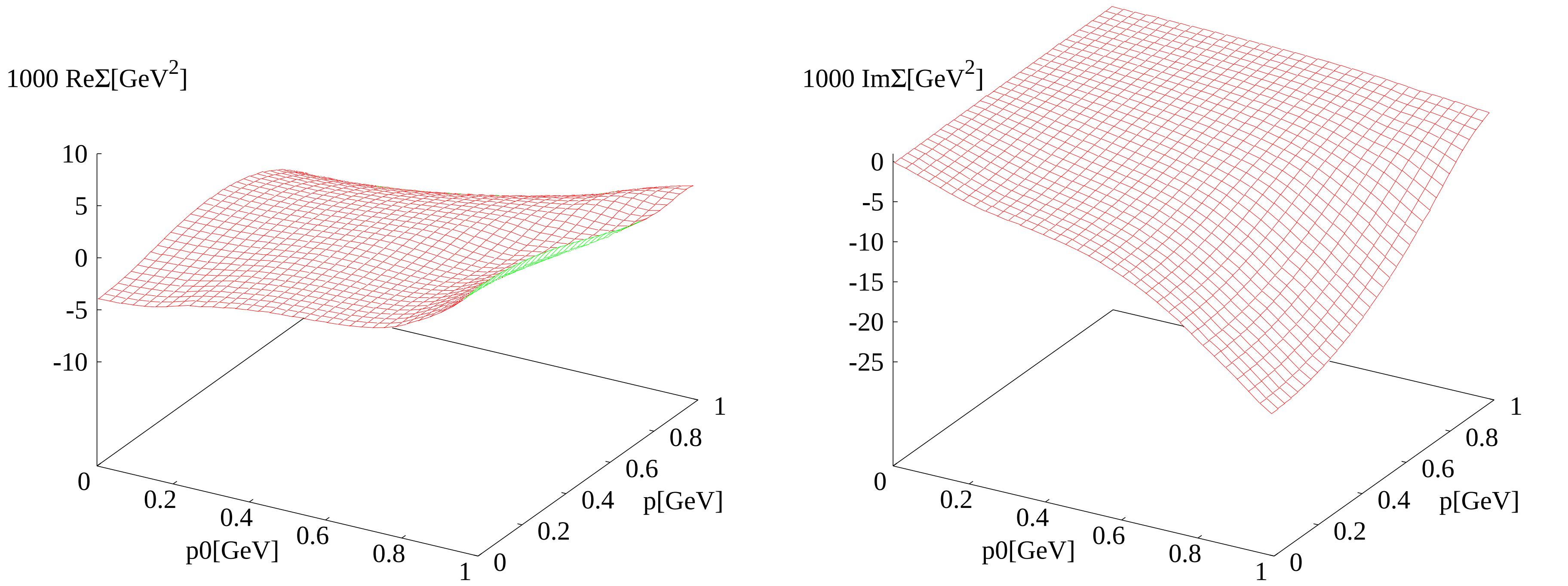
Results for the Vacuum Sunset Diagram





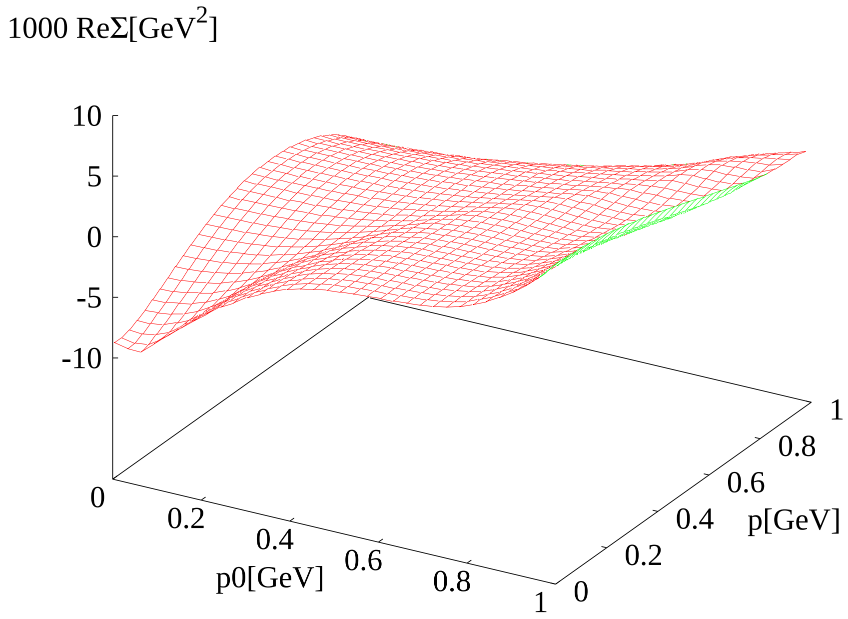
Vacuum: $m = 140\text{MeV}$, $\lambda = 50$

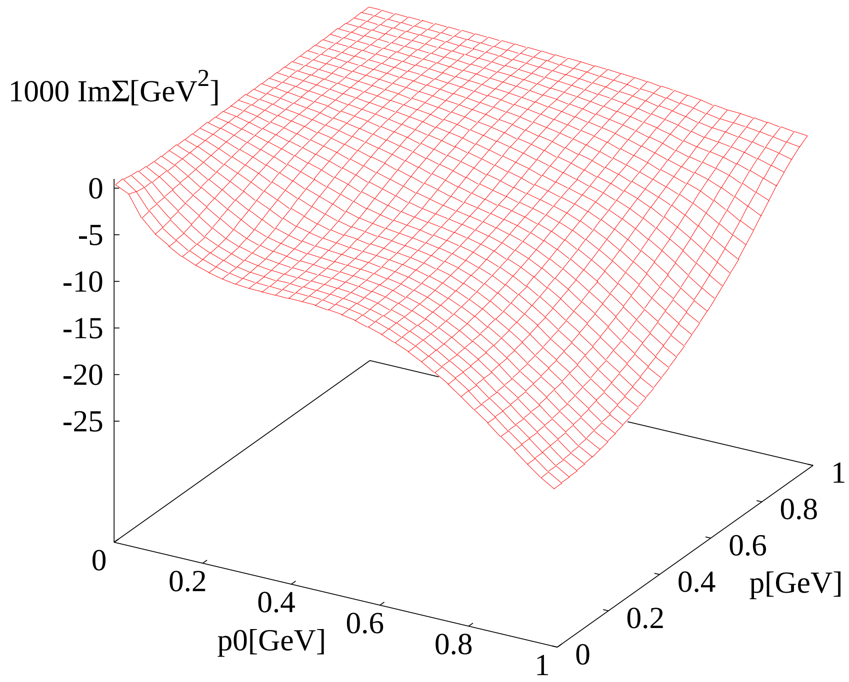
Perturbative Result for the Sunset Diagram at $T > 0$



$T = 100\text{MeV}$

Self-consistent Result for the Sunset Diagram at $T > 0$





$T = 100\text{MeV}$

The Analytic Green's Function

The imaginary part of the contour

- So far real-time formalism
- Entropy, pressure, mean energy,$\cdots$

- Analytic propagator
- Branch of analytic continuation of $G_{\text{Matsubara}}$

$$G_C(p_0, \vec{p}) = \int \frac{\mathrm{d}z'}{\pi} \frac{\rho(z', \vec{p})}{z' - p_0} \text{ with}$$
$$\forall z \in \mathbb{R} : \mathbb{R} \ni \rho(z, \vec{p}) = -\rho(-z, \vec{p}) = -\operatorname{Im} G_R(z, \vec{p})$$

- Causality structure of G_R and G_A

$$G_C(p_0 \pm \mathrm{i}0) = -G_{R/A}(p) \text{ for } p_0 \in \mathbb{R}$$

- Matsubara-propagator

$$G_M(\mathrm{i}\omega_n, \vec{p}) = G_C(\mathrm{i}\omega_n, \vec{p}) \text{ with } \omega_n = \frac{2\pi\mathrm{i}}{\beta} n = 2\pi\mathrm{i}nT, \; n \in \mathbb{Z}$$

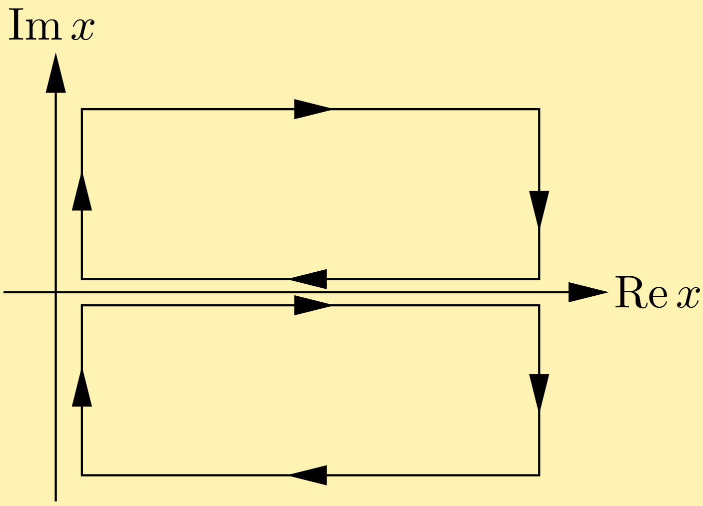
Matsubara Sums

Summing over Matsubara frequencies

- $F(z)$: analytic in an open strip around imaginary axis

$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}} F(\mathrm{i}\omega_n) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \mathrm{d}x[F(x)+F(-x)]\left[\frac{1}{2}+\frac{1}{\exp(\beta x)-1}\right]$$

- F(z): also analytic away from the real axis



$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}} F(\mathrm{i}\omega_n) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}x\frac{1}{2}[F(x-\mathrm{i}\epsilon\sigma(x))-F(x+\mathrm{i}\epsilon\sigma(x))]+$$
$$+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}x n(x)[F(x-\mathrm{i}\epsilon\sigma(x))-F(x+\mathrm{i}\epsilon\sigma(x))]$$

The Entropy

Thermodynamical properties

- Expectation values from thermal quantum field theory:
  $Z(\beta, V) = \operatorname{Tr}\exp(-\beta\mathbf{H})$
  $$\Rightarrow \varepsilon = \frac{1}{V}\langle\mathbf{H}\rangle = -\frac{1}{V}\partial_\beta \ln Z, \quad \frac{1}{V}\mathrm{d}(\beta\ln Z) = -\varepsilon\mathrm{d}\beta$$
- Define thermodynamical quantities:
  $$P = \frac{\ln Z}{\beta V}, \ s = \beta(P+\varepsilon) \Rightarrow \mathrm{d}P = s\mathrm{d}T$$
- Solution of the real time Φ-derived self-consistent equations
  $$\ln Z = \mathrm{i}\Gamma \Rightarrow P = \mathrm{i}\Gamma \Rightarrow s = \mathrm{i}\partial_T\Gamma$$
- Stationarity with respect to G_R: Need to derive only with respect to explicit temperature dependency

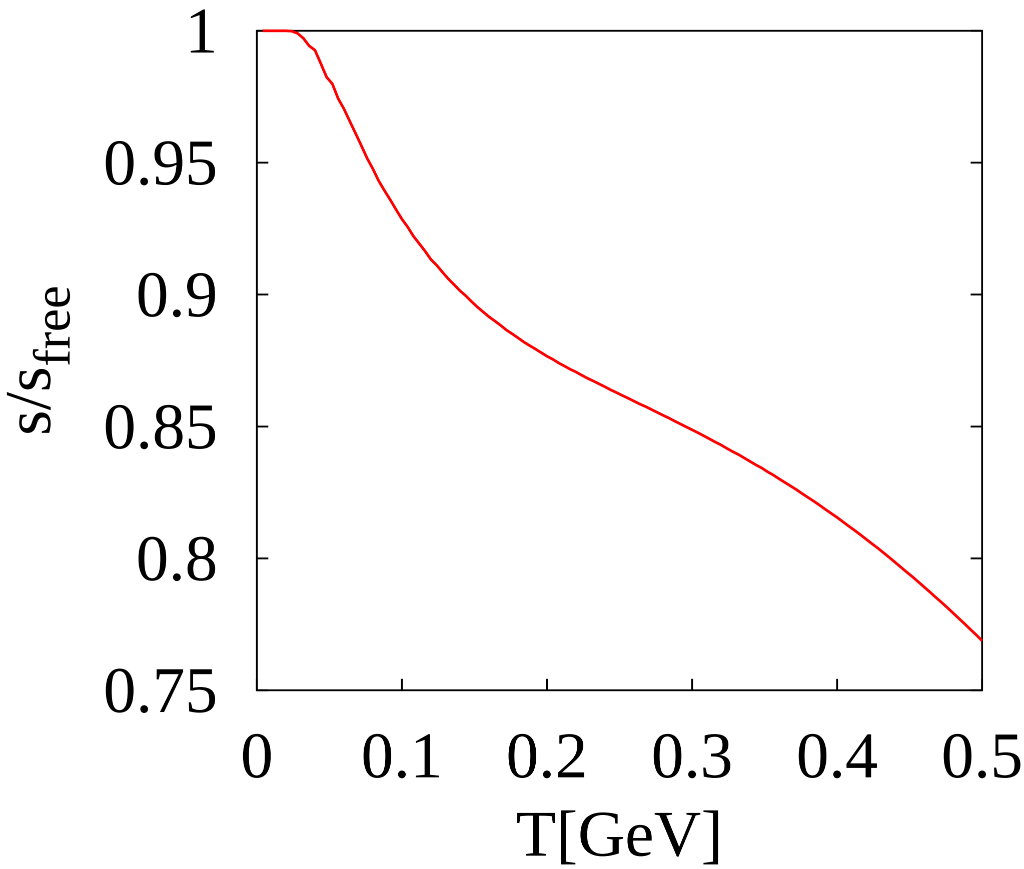
$$s = -2\int_{p_0>0}\frac{\mathrm{d}^4p}{(2\pi)^4}\partial_T n(p_0)\left\{\operatorname{Im}\ln[-G_R^{-1}(p)] + \operatorname{Im}(\Sigma_R G_R)\right\} +$$

$$+\,\mathrm{i}\left\{\frac{\delta\Phi[\varphi,G]}{\delta n}\right\}\Bigg|_{G_R,\varphi\ \text{fixed}}\partial_T n$$

- Especially for 2-point Φ-functionals

$$s = -2\int_{p_0>0}\frac{\mathrm{d}^4p}{(2\pi)^4}\partial_T n(p_0)\left\{\operatorname{Im}\ln[-G_R^{-1}(p)] + (\operatorname{Im}\Sigma_R)(\operatorname{Re}G_R)\right\}$$

Result for Tadpole Resummation



The entropy density for the free gas and for the selfconsistent tadpole resummation. The parameters were $m = 138$MeV and $\lambda = 50$.

Conclusion and outlook

Conclusions

- Self-consistent Φ-derivable models can be renormalized and solved numerically
- Applications for the consistent treatment of particles and resonances with finite mass widths possible
- Applicable as well for dynamics as for thermodynamical quantities
- Consistent schemes for transport equations for such particles and resonances

Outlook

- Problem with Goldstone's theorem and phase transitions (Linear Σ-model)
- Development of numerics for more complicated cases beyond two-point level (ladder summations)
- Big fundamental problem:
- Most important physical theories involve gauge fields

☞ Standardmodel

☞ (Phasetransitions in QCD and electro-weak Theory)

☞ Effective meson theories (vector dominance and dileptons)

- Does there exist a gauge-invariant scheme?