Contents

Selfconsistent Conserved Approximation for π- and ρ-Mesons

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- The investigated model for the ρ and π mesons
- Fit to data
- Selfconsistent approximations and renormalization
- Numerical Results in the vacuum and at finite Temperature
- Outlook

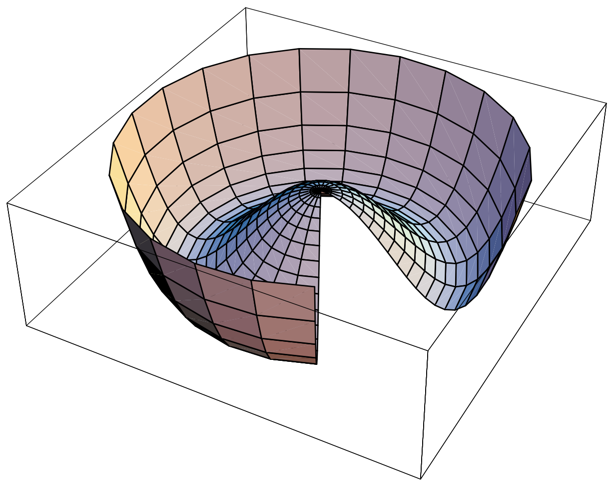
The Model

The ρ-mesons

- Renormalizable model for massive ρ-mesons $\Rightarrow$ Higgs-Kibble-formalism for Gauge theories
- Start with a $SU(2)$ duplett with gauged symmetry group

$$\mathscr{L}_1 = -\frac{1}{2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}(D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi)$$

- Mexican hat potential $V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2$



- Physical gauge (around the stable vacuum):
  - ρ-fields become massive $m_\rho^2 = g^2\mu^2/(4\lambda)$
  - Three Φ-degrees of freedom become ρ degrees of freedom
  - One Φ-degree of freedom gives a massive "Higgs-particle"

The Model

The Pions

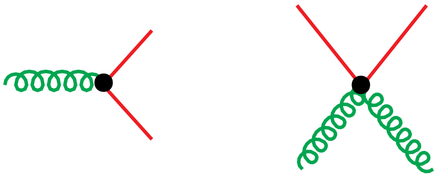
- Introduce Pions as adjoint representation, i.e., SO(3)-triplett

$$\mathscr{L}_2 = \frac{1}{2}(D_\mu \vec{\pi}) \cdot (D^\mu \vec{\pi}) - \frac{\lambda_2}{8}(\vec{\pi}^2)^2 - \frac{\lambda_3}{4}\vec{\pi}^2 \Phi^\dagger \Phi$$

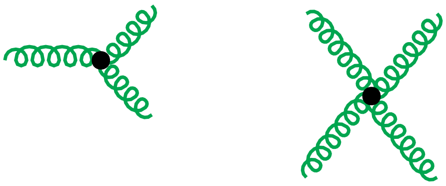
- Consistency condition:

$$m_\pi^2 = \frac{2m_\rho^2}{g}\lambda_3$$

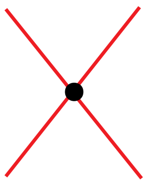
Unitary Gauge - Physical Vertices I



$\pi\rho$-interactions



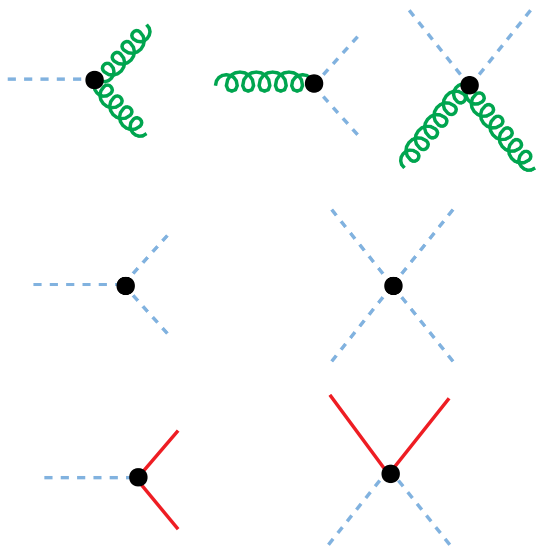
$\rho\rho$-interactions



π self interactions

The Model

Unitary Gauge - Physical Vertices II



ρ-Higgs-interactions

Higgs-Higgs-interactions

π-Higgs interactions

Remarks about Quantization

- Unitary gauge contains only physical dof. $\Rightarrow$ manifestly unitary
- To get renormalizable gauge $\Rightarrow$ Introducing R_ξ-gauges (van 't Hooft)
- R_ξ-gauge: manifestly renormalizble
- R_ξ-gauge: Faddeev-Popov-ghosts
- BRST-invariance $\Rightarrow$ S-Matrix gauge invariant
- R_ξ-gauge has unitary gauge as limit $\Rightarrow$ Renormalized theory also unitary

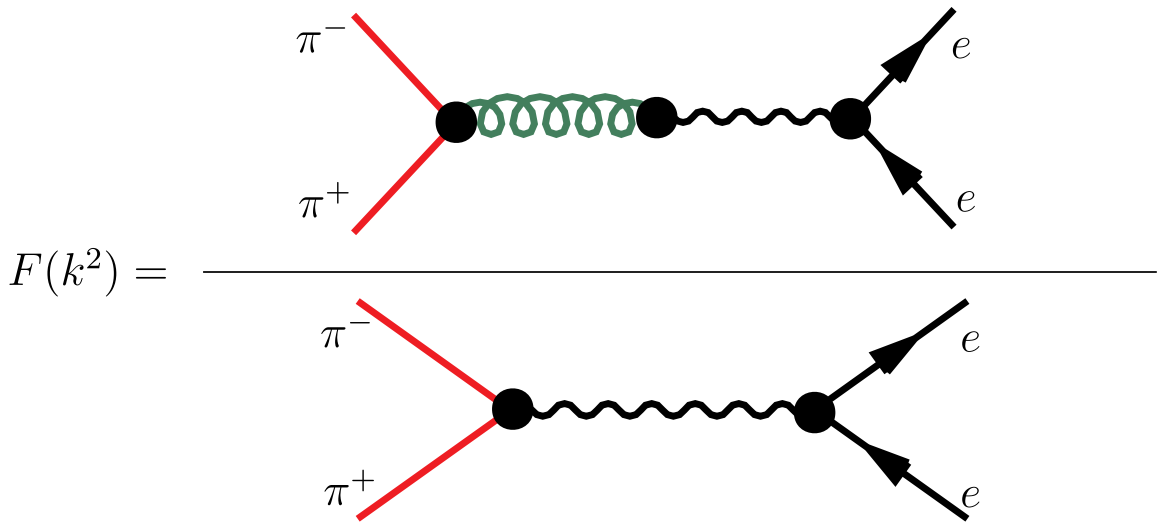
The Model

The Photon

- Extending the gauge group to $U(1) \times SU(2)$
- $U(1)$ unbroken $\Rightarrow$ One of the four gauge bosons remains massless $\Rightarrow$ photon
- Equations of Motion $\Rightarrow$ Pions couple to photons only through ρ $\Rightarrow$ Vector-Meson-Dominance

The Form Factor

- Electromagnetic Form Factor of the Pion:



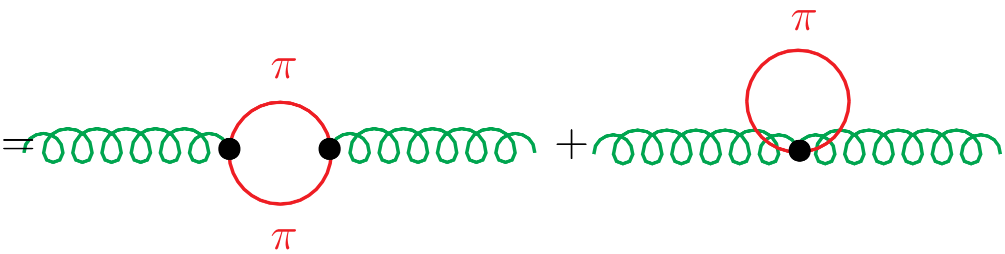
- Feynman rules: $\Gamma_{\rho\gamma} = \mathrm{i}\delta^{a3} M_\rho^2 e/g \Rightarrow$

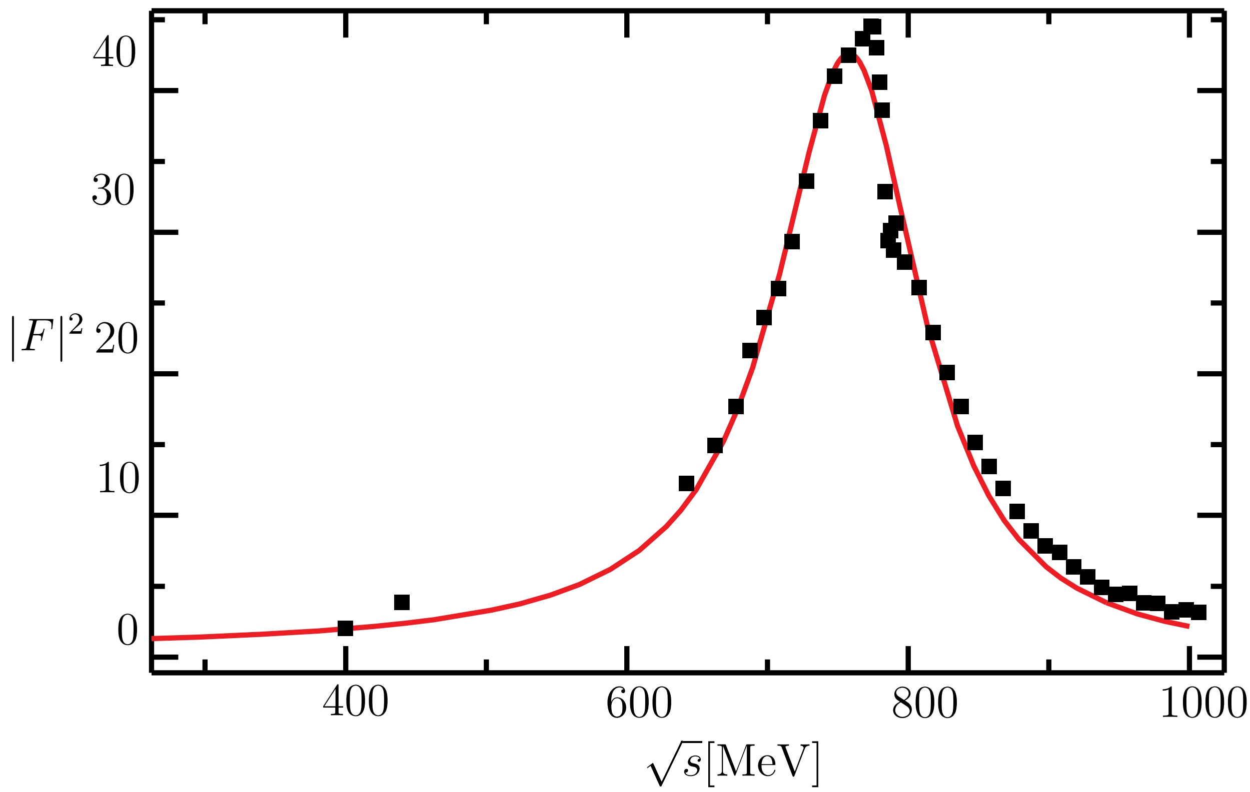
$$|F(s)|^2 = \frac{m_\rho^4}{[s - m_\rho^2 - \mathrm{Re}\,\Pi_\rho(s)]^2 + [\mathrm{Im}\,\Pi_\rho(s)]^2}$$

Fit of the parameters

Form factor and Phase Shift

- Using dimensional regularization and renormalization of the one-loop-self-energy diagrams

$-\mathrm{i}\Sigma_\rho =$ 



Data: Amendolia et al. Phys. Lett. **138B** (1984) 454
Barkov et al. Nucl. Phys. **B256** (1985) 365

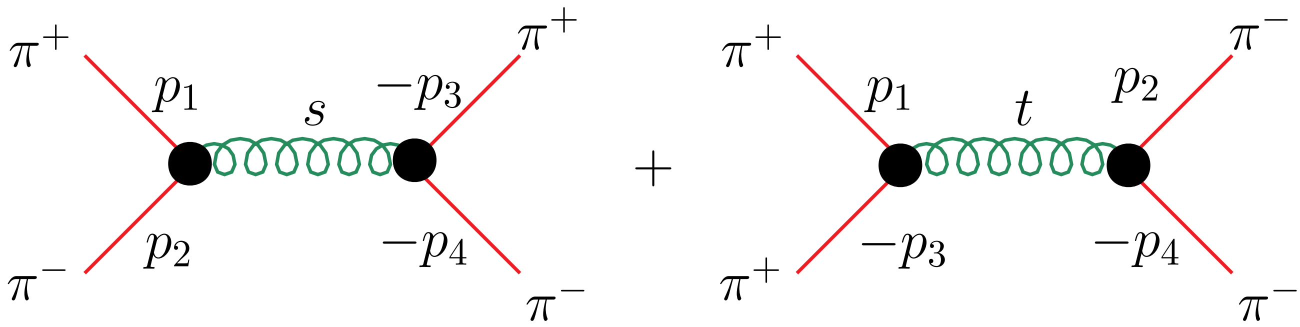
Fit of the parameters

Total $\pi^+\pi^-$ elastic cross-section

- Four π-vertex

$$\Gamma^{abcd}(p_1, \dots, p_4) = \begin{cases} & A(s,t,u)\delta_{ab}\delta_{cd} + \\ + & A(t,s,u)\delta_{ac}\delta_{bd} + \\ + & A(u,t,s)\delta_{ad}\delta_{bc} \end{cases}$$

- With the invariants $s = (p_1 + p_s)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^4$



- Feynman rules $\Rightarrow$ invariant transition amplitude:

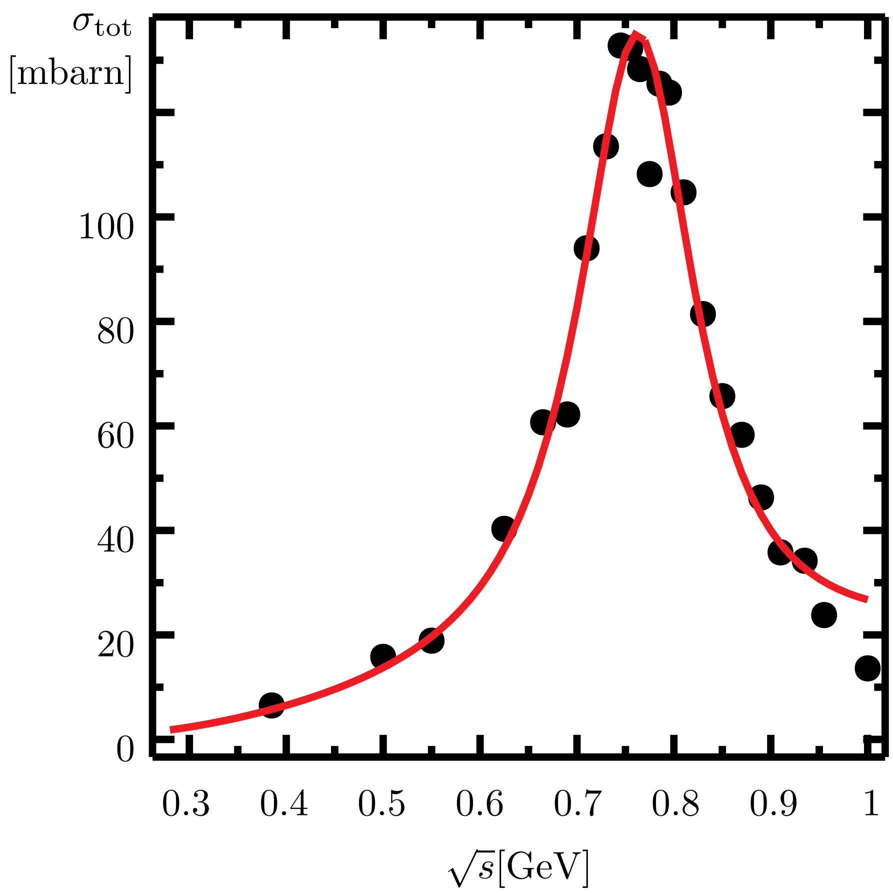
$$M_{fi}(s,t) = A(s,t,u) + A(t,s,u)|_{u=4m_\pi^2-s-t}$$

- Total cross section:

$$\sigma_{\text{tot}} = \frac{1}{64\pi} \frac{1}{s(s-4m_\pi)} \int_{4m_\pi^2-s}^{0} |M_{fi}(s,t)|^2$$

Fit of the parameters

- With the parameters from the fitting to phase-shift and form-factor:



- Data from: Forgatt, Petersen, Nucl. Phys. **B129** (1977) 89

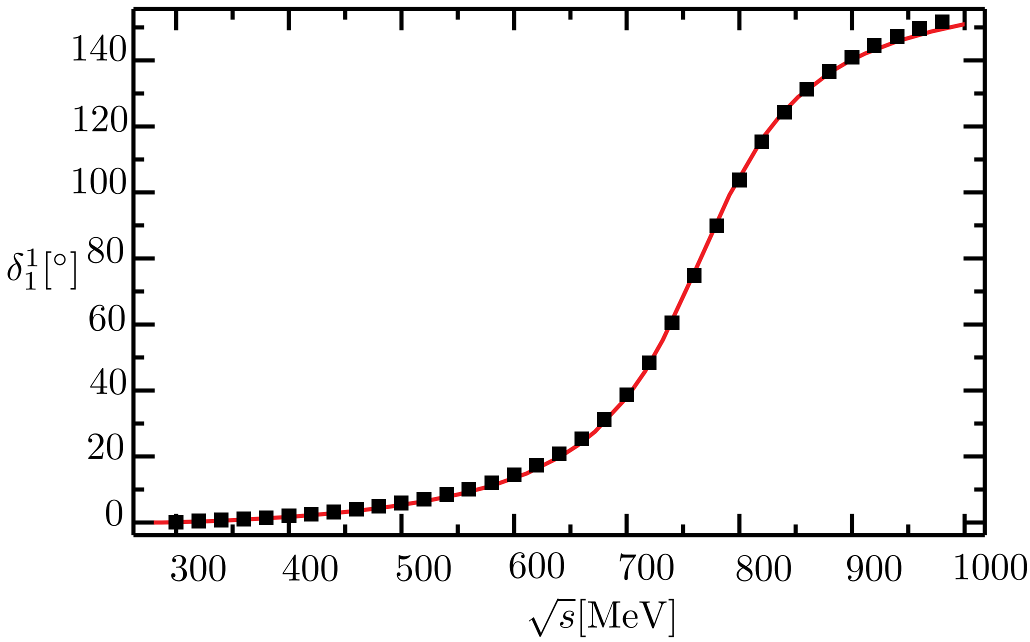
Fit of the parameters

Phaseshift in $t = 1$, $l = 1$-channel

- Projection to isospin $I = 1$: $M^{I=1} = A(s,u,t) - A(s,t,u)$
- From ρ-exchange ($s = E_{\mathrm{CM}}^2$, θ scattering angle in CM):

$$M^{I=1}(s,\theta) = 2g^2 \frac{(s - 4m_\pi^2)\cos\theta}{s - m_\rho^2 - \Pi_\rho(s)}$$

- Projection to angular momentum $l = 1$:
  $t_1^1(s) = \frac{1}{64\pi}\int_{-1}^{1} \mathrm{d}(\cos\theta)\cos\theta M^{I=1}(s,\theta)$
- Parametrization with phase shift $\delta_1^1(s) = \arccos\left[\frac{\mathrm{Re}\, G_\rho(s)}{|G_\rho(s)|}\right]$



Data: Frogatt, Petersen, Nucl. Phys. **B129** (1977) 89

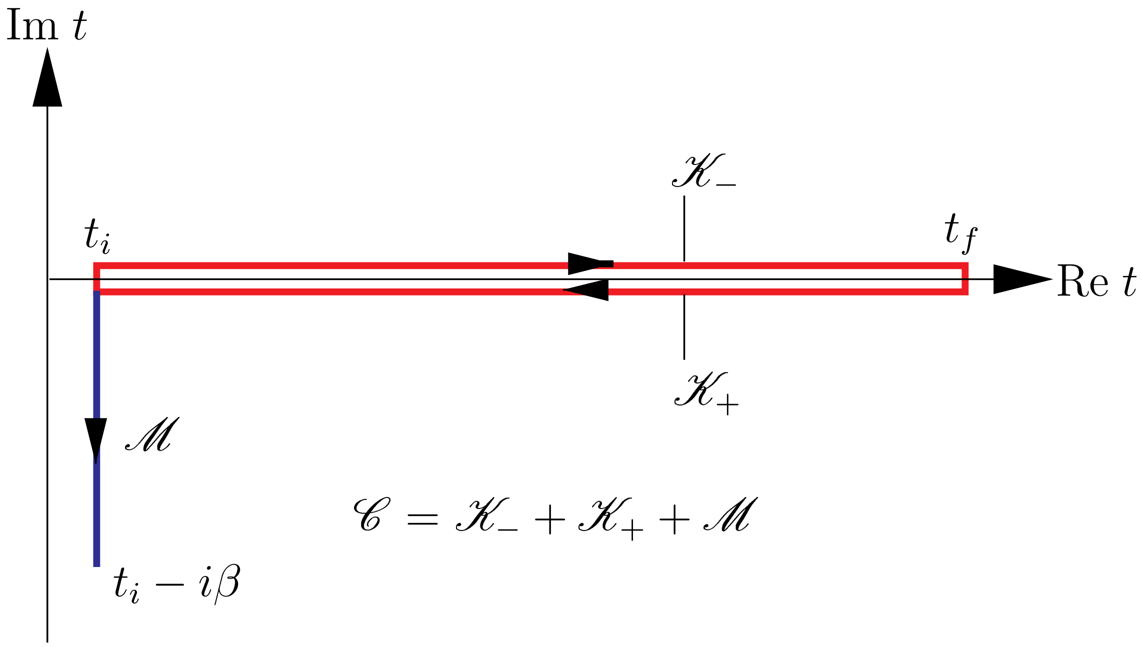
Selfconsistent approximations

The Functional

- Introduce a bilocal source term in addition to the local source term into the Z-functional:

$$Z[J,K] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\langle J_1\phi_1\right\rangle_1 + \frac{\mathrm{i}}{2}\left\langle K_{12}\phi_1\phi_2\right\rangle_{12}\right], \; W = -\mathrm{i}\ln Z$$

- S is the classical action of the field theory along the time axis (vacuum) or the Schwinger-Keldysh contour (thermal FT)



- Functional Legendre transformation wrt. both J and K:

$$\Gamma[\varphi,G] = W[J,K] - \left\langle \varphi_1 J_1\right\rangle_1 - \frac{1}{2}\left\langle (\varphi_1\varphi_2 + \mathrm{i}G_{12})K_{12}\right\rangle_{12}$$

$$\text{with } \varphi_1 = \frac{\delta W[J,K]}{\delta J_1} \text{ and}$$

$$G_{12} = -\frac{\delta^2 W[J,K]}{\delta J_1 \delta J_2} = -2\mathrm{i}\left(\frac{\delta W[J,K]}{\delta K_{12}} - \frac{1}{2}\varphi_1\varphi_2\right)$$

- Define $\Phi = \Gamma_2$ to be then 2PI vacuum diagrams with at least 1 loop:

$$\Gamma[\varphi,G] = S[\varphi] + \frac{\mathrm{i}}{2}\mathrm{Tr}\ln(DG^{-1}) + \frac{\mathrm{i}}{2}\left\langle \mathscr{D}_{12}^{-1}(G_{12} - \mathscr{D}_{12})\right\rangle_{12} + \Phi[\varphi,G]$$

Selfconsistent approximations

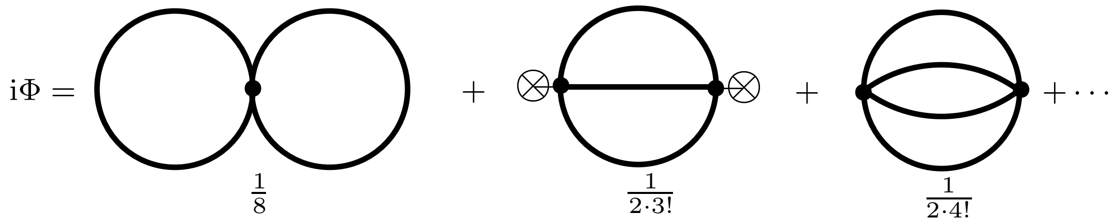
Diagramatics

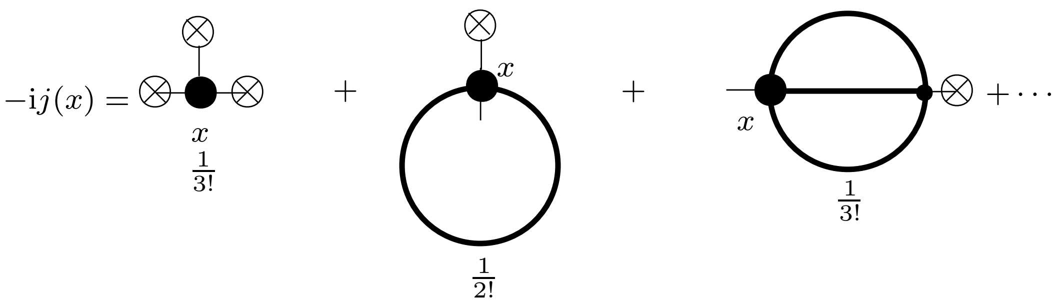
- Equations of motion: $J = K = 0$

$$\frac{\delta\Gamma[\varphi,G]}{\delta\varphi} = 0 \Leftrightarrow (-\Box - m^2)\varphi + \frac{\delta S_I}{\delta\varphi} + \frac{\mathrm{i}}{2}\left\langle \frac{\delta \mathscr{D}_{12}^{-1}}{\delta\varphi} G_{12} \right\rangle_{12} + \frac{\delta\Phi[\varphi,G]}{\delta\varphi} = 0$$

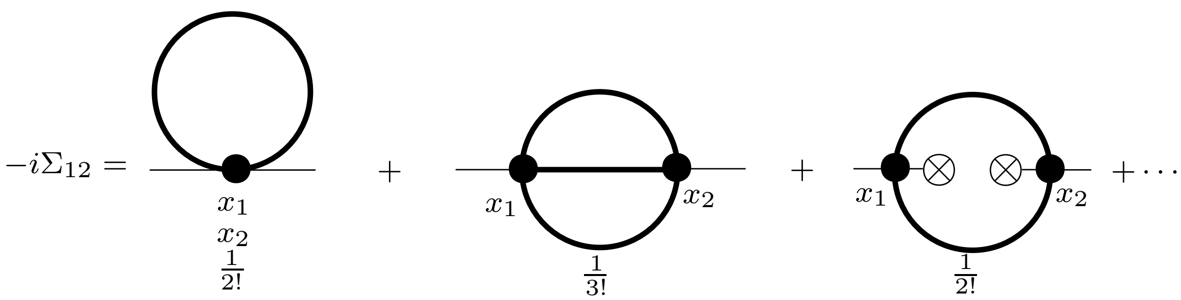
$$\frac{\delta\Gamma[\varphi,G]}{\delta G} = 0 \Leftrightarrow -\mathrm{i}\Sigma_{12} := -\mathrm{i}(\mathscr{D}_{12}^{-1} - G_{12}^{-1}) = 2\frac{\delta(\mathrm{i}\Phi[\varphi,G])}{\delta G}$$

- Simple example: ϕ^4-theory:





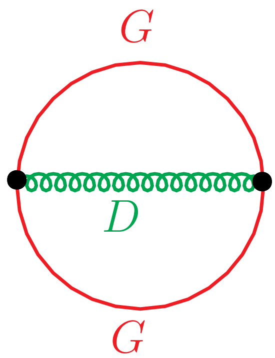
$$-(\Box + m^2)\varphi = j$$



Selfconsistent approximations

Generating functional

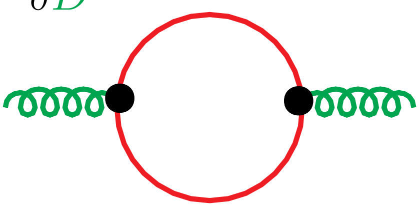
- $\Phi[G, D]$: sum over all 2PI closed diagrams with at least two loops

$$i\Phi[G, D] =$$ 

$$+ \cdots$$

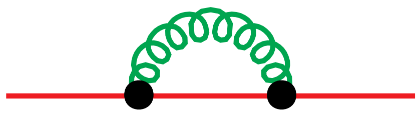
- Variation with respect to Green's functions $\Rightarrow$ self energies fulfilling Dyson's equations

$$\frac{\delta i\Phi}{\delta D} = -i\Pi_\rho =$$



$$\Pi_\rho = D_0^{-1} - D^{-1}$$

$$\frac{\delta i\Phi}{\delta G} = -i\Sigma_\pi =$$



$$\Sigma_\pi = G_0^{-1} - G^{-1}$$

- Sum up to a certain loop order $\Rightarrow$ Selfconsistent effective approximation
- Respects all conservation laws basing on global symmetries
- In thermal field theory: Thermodynamically consistent approximation

Renormalization

Renormalizing the selfconsistent approximation

- Can be seen as resummation of all self energy insertions $\Rightarrow$ Infinities to all orders
- Renormalizable theory $\Rightarrow$ finite by renormalizing parameters already present in Lagrangian
- Physical renormalization conditions

$$\Sigma_\pi(m_\pi^2) = \partial_s \Sigma_\pi(m_\pi^2) = 0, \; \Pi_\rho(0) = \partial_s \Pi_\rho(0) = 0$$

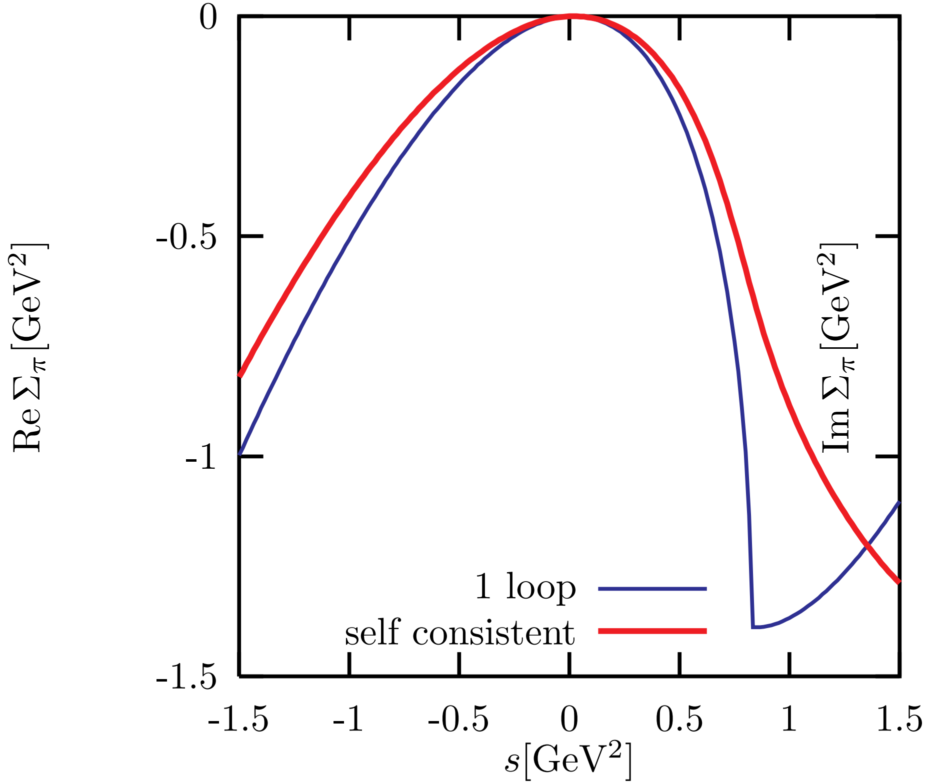
- Analytical properties of Green's functions

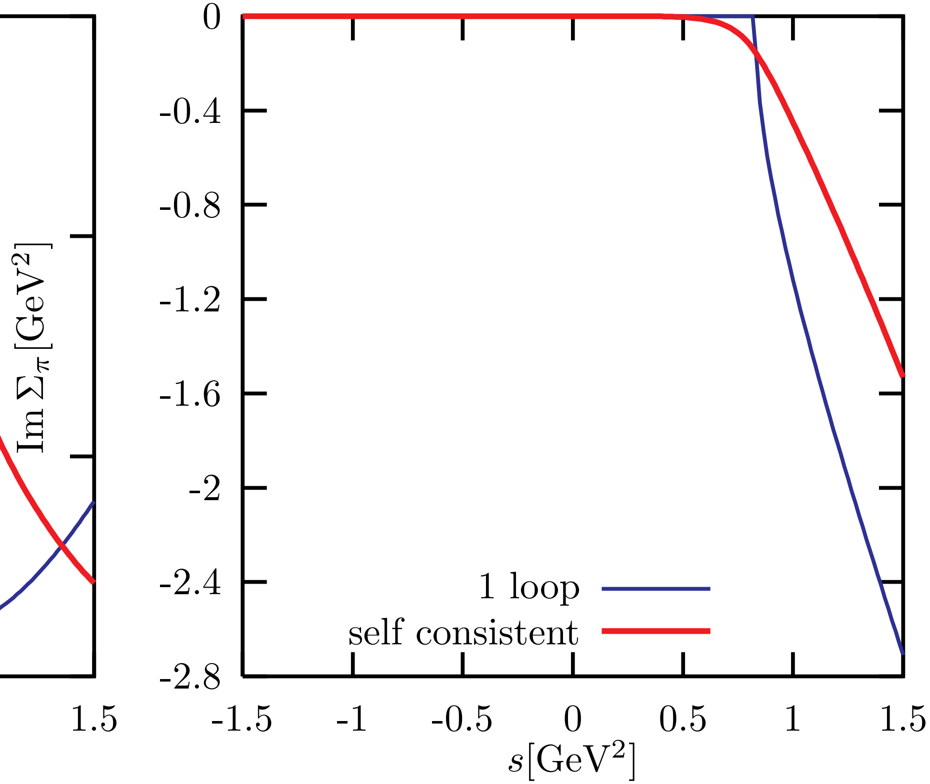
$$G(s) = \frac{1}{\pi} \int_0^\infty \mathrm{d}m^2 \Delta(m^2, s) A(m^2) \text{ with } A(s) = -\operatorname{Im} G(s)$$

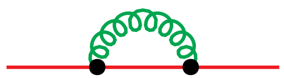
- $\Delta(m^2, s)$: Feynman-propagator $\Rightarrow$ integral kernels $\Rightarrow$ can be renormalized using standard techniques
- self consistent finite set of coupled integral equations solvable numerically by iteration
- Tadpole in vacuum absorbed into mass renormalization

Results in vacuum

The π-Self-Energy

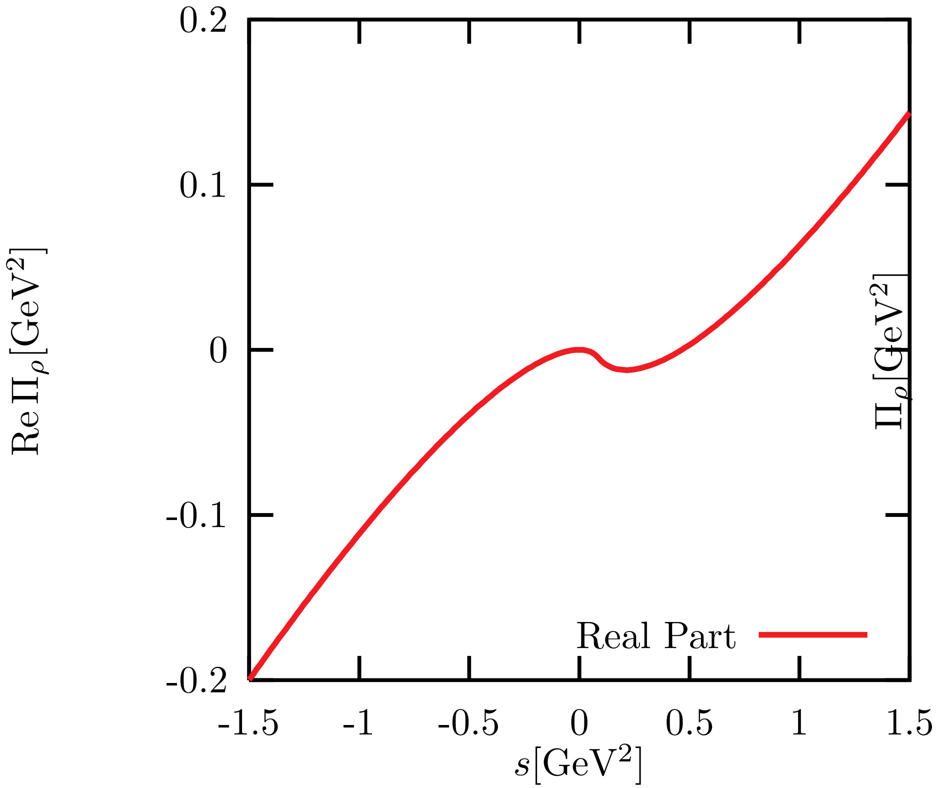


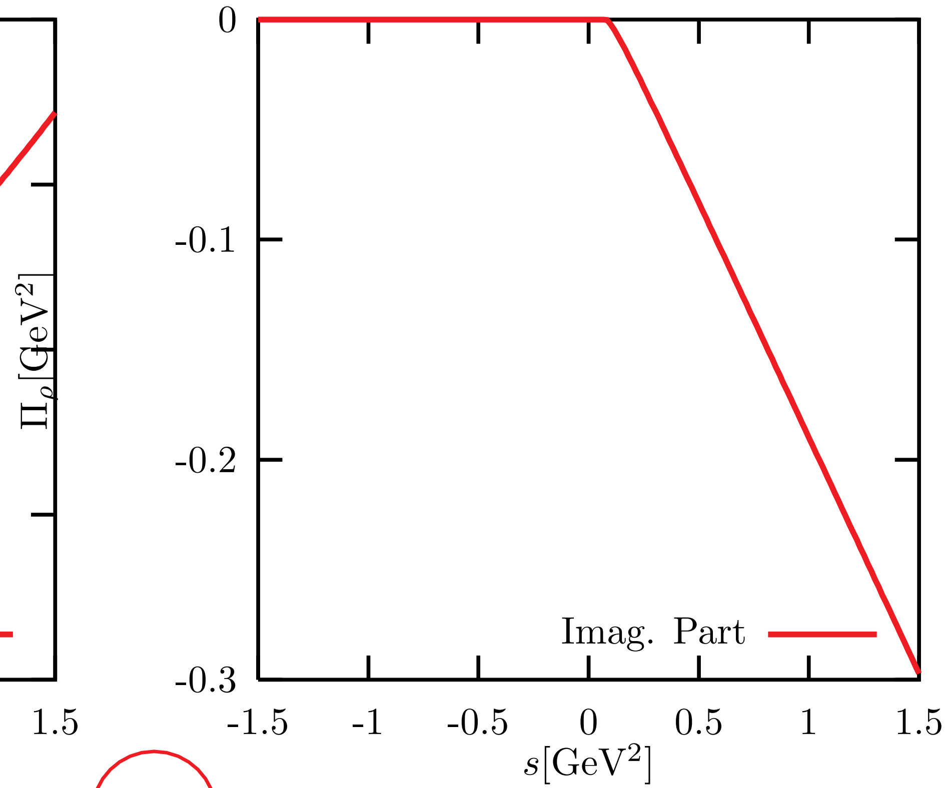


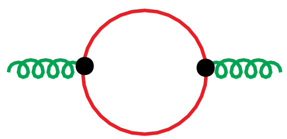
$-\mathrm{i}\Sigma_\pi =$ 

Result in vacuum

The ρ-Self-Energy



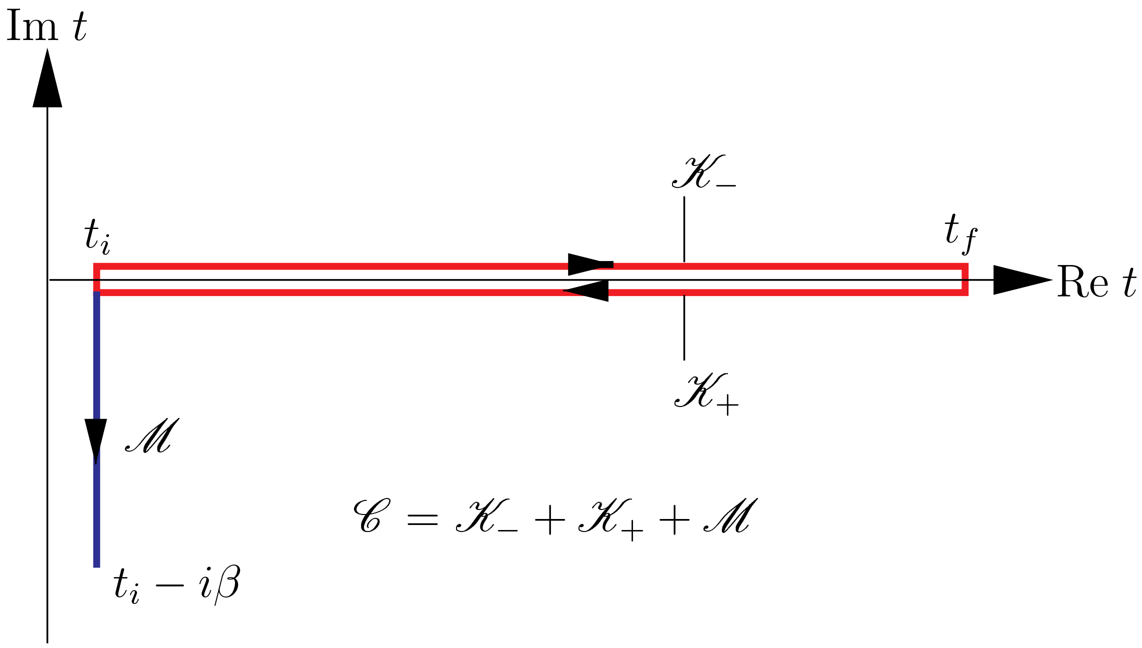


$-\mathrm{i}\Pi_\rho =$ 

Finite Temperature

Quantum Field Theory at finite Temperature

- Using the modified Schwinger Keldysh contour for equilibrium



- Timeordering in vacuum → contour ordering
- Path integral formalism: Generating functional Z factorizes in real time and imaginary time part.
- Calculate $\langle O \rangle = \mathrm{Tr}[\exp(-\beta \underset{\sim}{H}) \underset{\sim}{O}]$
- Wick's theorem ⇒ Path ordered Green's functions → Matrix Formalism
- Trace ⇒ (Anti-)Periodicity of fields → KMS condition

Finite Temperature

Analytic Properties of Green's Functions

- The 2-point Green's functions can be expressed in terms of the spectral function:

$$\mathrm{i}G^{--}(p) = \int_0^\infty \frac{\mathrm{d}k_0}{\pi} \frac{2\mathrm{i}k_0 A(p_0,\vec{p})}{p_0^2 - k_0^2 + \mathrm{i}\epsilon} + 2n(p_0)A(p^2,\vec{p}),$$

$$\mathrm{i}G^{++}(p) = -\int_0^\infty \frac{\mathrm{d}k_0}{\pi} \frac{2\mathrm{i}k_0 A(p_0,\vec{p})}{p_0^2 - k_0^2 - \mathrm{i}\epsilon} + 2n(p_0)A(p^2,\vec{p}),$$

$$\mathrm{i}G^{+-}(p) = 2[\Theta(p_0) + n(p_0)]A(p_0,\vec{p}) = 2[1+f(l_0)]\tilde{A}(p),$$

$$\mathrm{i}G^{-+}(p) = 2[\Theta(-p_0) + n(p_0)]A(p_0,\vec{p}) = 2f(l_0)\tilde{A}(p).$$

with $\tilde{A}(p) = -\operatorname{Im} G_R(p) = \operatorname{sign} p_0 A(p)$

and $f(x) = \dfrac{1}{\exp(\beta x) - 1}$ $n(x) = f(|x|)$

- Feynman rules for imaginary part:

$$\operatorname{Im} G_R = \frac{G^{+-} - G^{-+}}{2\mathrm{i}} \tag{1}$$

- Self energies and Dyson equation

$$G_R(p) = \frac{1}{p^2 - m^2 - \Sigma_R}, \quad \operatorname{Im}\Sigma_R = \frac{\Sigma^{-+} - \Sigma^{+-}}{2\mathrm{i}}$$

- Causality ⇒ Kramers-Kronig-Relations (Dispersion relations):

$$f(z) = \int_{-\infty}^\infty \frac{\mathrm{d}z'}{\pi} \frac{\operatorname{Im} f(z')}{(z'-z)(z'-z_r)^n} + \sum_{k=0}^{n=1} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k.$$

- Crucial: Subtractions ONLY in vacuum parts of the self energies!

Π and ρ at finite Tempererature

The selfconsistent equations

- Breaking of Lorentz invariance due to temperature:

$$\Pi_{\mu\nu} = -\Pi_T \Theta^T_{\mu\nu} - \Pi_L \Theta^L_{\mu\nu} \text{ with:}$$

$$\Theta_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2},$$

$$\Theta^T_{\mu\nu} = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0 \\ -\delta_{ij} + \frac{p_i p_j}{\vec{p}^2} & \text{for } \mu, \nu \in \{1,2,3\} \end{cases},$$

$$\Theta^L_{\mu\nu} = \Theta_{\mu\nu} - \Theta^T_{\mu\nu}$$

- Dyson equation for transverse gauge (Landau gauge):

$$G^\rho_{\mu\nu} = -\frac{\Theta^L_{\mu\nu}}{p^2 - m^2 - \Pi_L} - \frac{\Theta^T_{\mu\nu}}{p^2 - m^2 - \Pi_T}$$

- Calculate iteratively: Imaginary part of self energies (finite!):

$$\mathrm{Im}\,\Pi^L_{\mu\nu}(p) = -\frac{g^2}{2\pi^4} \int \mathrm{d}^4 l \frac{[p_0(\vec{l}\vec{p}) - l_0 \vec{p}]^2}{\vec{p}^2 p^2} [f(l_0) - f(l_0 + p_0)] A_\pi(l+p) A_\pi(l)$$

$$\mathrm{Im}\,\Pi^T_{\mu\nu}(p) = -\frac{g^2}{4\pi^4} \int \mathrm{d}^4 l \frac{\vec{l}^2 \vec{p}^2 - (\vec{p}\vec{l})^2}{\vec{p}^2} [f(l_0) - f(l_0 + p_0)] A_\pi(l+p) A_\pi(l)$$

$$\mathrm{Im}\,\Sigma(p)_\pi = -\frac{g^2}{\pi^4} \int \mathrm{d}^4 l [f(l_0) - f(l_0 + p_0)](2p_\mu + l_\mu)(2p_\nu + l_\nu) \times$$
$$\times \left[\Theta^{\mu\nu}_L(l) A_{\rho L}(l) + \Theta^{\mu\nu}_T(l) A_{\rho T}(l)\right] A_\pi(l+p)$$

- Calculate real parts for the temperature part with a dispersion relation without subtractions

Dilepton Rate

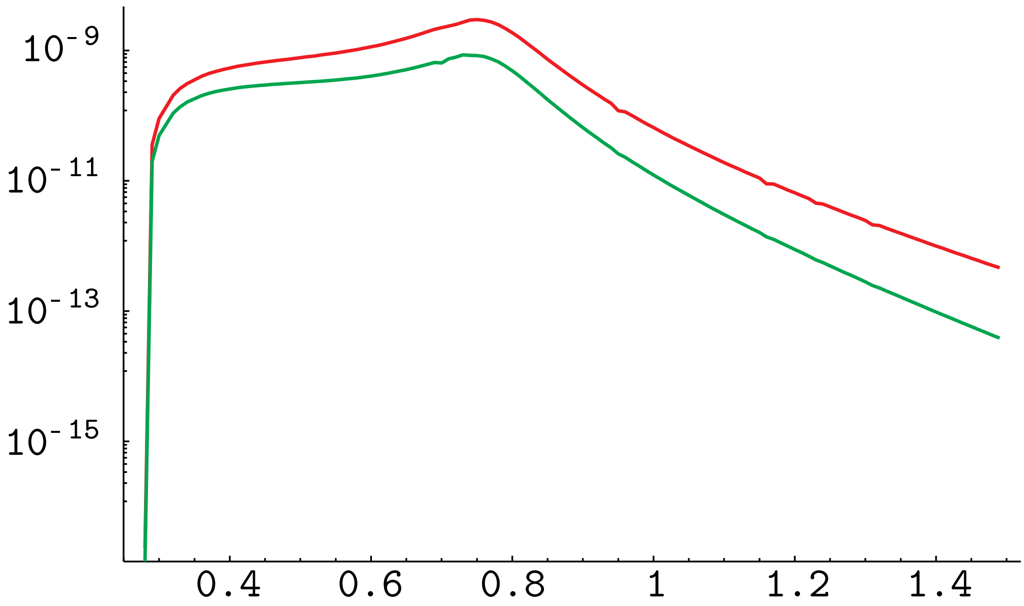
The Dilepton Rate

- Kadanoff-Baym-Equations: Exact result for strong coupling:

$$\left.\frac{\mathrm{d}^4 R}{\mathrm{d}\sqrt{s}\mathrm{d}P^3}\right|_{\vec{P}=0} = \frac{2\alpha^2}{(2\pi)^3}\frac{m_\rho^2}{g^2}\frac{1}{s}A_\rho(\sqrt{s},0)f_B(\sqrt{s})$$

- Dilepton Production Rate

$$\frac{\mathrm{d}^4 R}{\mathrm{d}\sqrt{s}\mathrm{d}^3\vec{P}}[\mathrm{GeV}^{-3}]$$



$$\sqrt{s}[\mathrm{GeV}]$$

- $T = 150\mathrm{MeV}$, $200\mathrm{MeV}$

Outlook

Work to do

- Exploit non-abelian part of the ρ-interaction
- Include other particles (A_1, ω, N, Δ, $\ldots$)
- Question of theoretical interest: Is it possible to extend the selfconsistent approximation scheme to a gauge invariant one?