Advanced quantum mechanics

SS 2019 – Prof. Dr. Marc Wagner

Organization: Room GSC 0|21

Christian Reisinger: reisinger@th.physik.uni-frankfurt.de

Exercise sheet 4

To be handed in 16.05.19. To be discussed in the week of 20.05.19.

09.05.19

Exercise 1 [*Preparation of a projectile wave function*] (4 pts.)

In the lecture, the wave function of a projectile for a scattering experiment at time t_0 before the scattering process was written as a wave packet

$$\psi(\mathbf{r}, t_0) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \; a(\mathbf{k}) e^{+i\mathbf{k}\mathbf{r}}. \tag{1}$$

Construct a suitable function $a(\mathbf{k})$ (a wave function in momentum space), such that the projectile has the properties discussed in the lecture. In particular, the projectile should have momentum $\approx \mathbf{k}_0$, at time t_0 it has to be sufficiently far away from the potential $V(\mathbf{r})$ (centered at the origin), and at a later time it has to collide with the potential representing the target.

Exercise 2 [*Green's function for the stationary free Schrödinger equation*] (7+2+4+3=16 pts.)

The free Schrödinger equation can be written in the form

$$\left(\Delta + k^2\right) \psi(\mathbf{r}) = 0. \tag{2}$$

In the lecture, the Green's function of the differential operator $\Delta + k^2$ was given:

$$G_+(\mathbf{r}) = -\frac{1}{4\pi} \frac{e^{+ikr}}{r}. \tag{3}$$

(a) Perform a detailed calculation starting from the defining equation

$$\left(\Delta + k^2\right) G_+(\mathbf{r}) = \delta(\mathbf{r}) \tag{4}$$

and show that eq. (3) is indeed a Green's function of $\Delta + k^2$. For the calculation, it is convenient to switch to Fourier space and then use the residue theorem to do the integration. Is the result of the integration unique? Discuss the positions of the poles relative to your path of integration.

(b) Show that eq. (3) corresponds to particle stream of an outgoing spherical wave, by calculating the corresponding current.

(c) Choose a different path of integration, such that you obtain a Green's function $G_-(\mathbf{r})$, which describes a particle stream of an incoming spherical wave. Verify your result, by again calculating the corresponding current.

(d) In section 2.2 of the lecture, one of the important results

$$\psi(\mathbf{r},t) \approx \psi_{\text{free}}(\mathbf{r},t) + \int \mathrm{d}^3k\ a(\mathbf{k}) \frac{e^{-iE(\mathbf{k})(t-t_0)/\hbar + ikr}}{r} f_{\mathbf{k}}(\vartheta,\varphi), \qquad (5)$$

for $|\mathbf{r}| \gg R$ (R denotes the range of the potential), was obtained by using $G_+(\mathbf{r})$. One might expect that the calculation in section 2.2 can be done with $G_-(\mathbf{r})$ in an analogous way. Go through the steps of the derivation in detail, and find out at which point the use of $G_+(\mathbf{r})$ is essential. In particular discuss what would be the problems when using $G_-(\mathbf{r})$.