Advanced quantum mechanics

SS 2019 – Prof. Dr. Marc Wagner

Organization: Room GSC 0|21
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Exercise sheet 9

To be handed in before 20.06.19, 11:00 by e-mail or in office 2.107.
To be discussed in the week of 24.06.19.
13.06.19

Exercise 1 [*Dirac equation and γ-matrices*] (3+2+4=9 pts.)

(a) Prove the important property of the γ-matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

by using the properties of the Pauli matrices and their product.

(b) Show that the components of a solution $\psi(x)$ of the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ also fulfill the Klein-Gordon equation, i.e. that

$$\left(\Box + m^2\right)\psi(x) = 0. \tag{2}$$

(c) In the lecture the γ-matrices in the standard representation were introduced. There are other equivalent useful representations, e.g. the so called "Weyl" or "chiral" representation,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \ , \ \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \tag{3}$$

where σ^j are the Pauli matrices.

- Show that a change of the representation can be described by a linear transformation of the four spin components of ψ by explicitly constructing this linear transformation between the standard and Weyl representation.
- The standard representation is useful for non-relativistic particles (often heavy particles), because two of the four spin components of ψ nearly vanish. For which kind of particles can a similar statement be made in the chiral representation, i.e. in which case does the Dirac equation decouple in two 2×2 matrix equations?

Exercise 2 [*Training with γ-matrices*] (3+3+3+2=11 pts.)

(a) Show that an infinitesimal Lorentz transformation (infinitesimal angle or boost-velocity) has the form

$$\Lambda^\mu{}_\nu = \eta^\mu{}_\nu + \epsilon^\mu{}_\nu, \tag{4}$$

with $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$.

(b) Show that infinitesimal Lorentz transformations $S(\Lambda)$ can be written as

$$S(\Lambda) = 1 - \frac{i}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}, \tag{5}$$

with $\sigma_{\mu\nu} \equiv i[\gamma_\mu, \gamma_\nu]/2$, by verifying that this expression is a solution of the defining equation of $S(\Lambda)$,

$$\gamma^\nu = S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu. \tag{6}$$

(c) Show that, for a finite Lorentz transformation, the corresponding transformation matrix for spinors is

$$S(\Lambda) = \exp\left(-\frac{i}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}\right). \tag{7}$$

Relate the entries of $\epsilon^{\mu\nu}$ to (i) the relative velocity v of a boost in x-direction and (ii) to the angle α of a rotation around the x-axis.

(d) Show that the bilinear $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector under Lorentz transformations.