Advanced quantum mechanics

SS 2019 – Prof. Dr. Marc Wagner

Organization: Room GSC 0|21
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Exercise sheet 10

To be handed in 27.06.19 before the lecture.
To be discussed in the week of 01.07.19.
20.06.19

Exercise 1 [*Pauli equation, auxiliary calculations*] (5+4=9 pts.)

Show the following relations, which were used in the lecture, when deriving the Pauli equation

(a)

$$\left(\sum_j \sigma_j \left(-i\partial_j - eA^j(\mathbf{r},t)\right)\right)^2 = \left(-i\nabla - e\mathbf{A}(\mathbf{r},t)\right)^2 - e\vec{\sigma}\mathbf{B}(\mathbf{r},t). \qquad (1)$$

(b) For constant magnetic field and $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$,

$$\left(-i\nabla - e\mathbf{A}(\mathbf{r})\right)^2 = \mathbf{p}^2 - e\mathbf{L}\mathbf{B} + e^2(\mathbf{A}(\mathbf{r}))^2, \qquad (2)$$

with $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{p} = -i\nabla$.

Exercise 2 [*Relativistic treatment of a pion in a Coulomb potential*] (1+2+3+1+4=11 pts.)

A pion π^- is a quark-antiquark pair $d\bar{u}$, i.e. a boson with negative electric charge $-e$. A quantum mechanical treatment of such a pion in a Coulomb potential of an atomic nucleus with proton number Z is possible with the Klein-Gordon equation.

(a) Write down the Klein-Gordon equation of a pion in an electromagnetic field with the 4-potential $A^\mu = (\phi, \mathbf{A})$.

(b) Consider a time-independent 4-potential $A^\mu(\mathbf{r})$ and solutions with positive energy. Use the technique of separation of variables to derive the corresponding stationary Klein-Gordon equation, which is the relativistic analog to the stationary Schrödinger equation.

(c) Consider a time-independent and rotationally symmetric 4-potential $A^\mu(r)$. Use again the technique of separation of variables to obtain an ordinary differential equation in the radial coordinate r from the stationary Klein-Gordon equation you derived in (b).

(d) For a Coulomb potential $\phi = -Ze/r$, $\mathbf{A}(r) = 0$, show that the differential equation for the radial part $R(r)$ of the solution of the Klein-Gordon equation from (c) has the form

$$\left(-\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{l(l+1) - Z^2e^4}{r^2} - \frac{2Ze^2E}{r} - (E^2 - m^2)\right) R(r) = 0. \qquad (3)$$

(e) The solution of this equation gives energies

$$E_{nl} = m\left(1 + \frac{Z^2e^4}{(n - (l+1/2) + \sqrt{(l+1/2)^2 - Z^2e^4})^2}\right)^{-1/2}, \qquad (4)$$

with $n = 1, 2, \ldots$ and $l = 0, 1, \ldots, n-1$ (cf. e.g. F. Schwabl, "Quantenmechanik für Fortgeschrittene (QMII)", Springer, section 8.1.2). Identify a small dimensionless parameter and expand E_{nl} as a series in terms of this parameter. Compare your result to the one obtained from a non-relativistic calculation using the Schrödinger equation, which you can find in the literature. What is the leading order relativistic correction that follows from your calculation? Make a qualitative plot of the energy spectrum for small n and discuss, which degeneracies disappear due to relativistic corrections.