Advanced quantum mechanics

SS 2019 – Prof. Dr. Marc Wagner

Organization: Room GSC 0|21

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Exercise sheet 12

To be handed in 11.07.19 before the lecture.
To be discussed in the week of 15.07.19.
04.07.19

Exercise 1 [*$U(1)$ gauge theory*] (5+5=10 pts.)

In the lecture it was motivated why the group $U(1)$ is used to describe electrodynamics. In this context the coupled equations

$$\begin{aligned} &\left((\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) + m^2\right)\phi = 0 \\ &\partial_\mu F^{\mu\nu} = 4\pi ie\left(\phi^*((\partial^\nu + ieA^\nu)\phi) - ((\partial^\nu + ieA^\nu)\phi)^*\phi\right) \end{aligned} \tag{1}$$

were considered.

(a) Show that both equations are invariant under a local $U(1)$ transformation

$$\phi'(x) = g(x)\phi(x) \quad , \quad A'^\mu(x) = g(x)\left(A^\mu(x) + \frac{i}{e}\partial^\mu\right)g^{-1}(x), \tag{2}$$

with $g(x) \in U(1)$.

(b) Besides $\partial_\mu F^{\mu\nu}$, there is only one other term in which A^μ appears linearly and ∂_μ quadratically, that can be used to write down an equation that is gauge invariant and invariant under Poincare transformations. Find this term and provide the corresponding familiar equation. Does this equation have any implications for A^μ? If not, why is this equation still useful?

Exercise 2 [*Transformation of Dirac spinors in the standard and chiral representation*] (5+2+3=10 pts.)

In the lecture, the behavior of a Dirac spinor with 4 components under Poincare transformations was derived in different ways. The transformed spinor is

$$\psi' = \exp\left(-\frac{i}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}\right)\psi \quad , \quad \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{3}$$

in the standard representation and

$$\tilde{\psi}' = \begin{pmatrix} e^{+i\alpha_j\sigma_j/2+\eta_j\sigma_j/2} & 0 \\ 0 & e^{+i\alpha_j\sigma_j/2-\eta_j\sigma_j/2} \end{pmatrix}\tilde{\psi}, \tag{4}$$

with angles α_j and rapidities η_j, in the chiral representation. Show that the two equations are equivalent by following these steps:

(a) Show that eq. (3) is also valid for the chiral representation, i.e. when replacing $\psi \to \tilde{\psi} = (1 + \gamma^0\gamma^5)\psi/\sqrt{2}$ and $\gamma^\mu \to \tilde{\gamma}^\mu = (1 + \gamma^0\gamma^5)\gamma^\mu(1 - \gamma^0\gamma^5)/2$. *Hint: Assume that eq.* (3) *is valid for the chiral representation and derive from that eq.* (3) *for the standard representation.*

(b) Relate the parameters $\epsilon^{\mu\nu}$ to α_j, η_j, e.g. by considering Lorentz transformations of 4-vectors.

(c) Use your results to show that in the chiral representation, the transformation of a Dirac spinor is given by eq. (4).