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Exercise sheet 2

To be discussed on 27, 30 April and 04, 07 May

Exercise 1 [*Third order Runge-Kutta method*]

Consider the differential equation $\dot{y}(t) = f(t, y)$, with y and f being at least 4-times differentiable. The Runge-Kutta method is a numerical procedure to iteratively obtain an approximate solution for $y(t)$. Show that for a given time t the third-order Runge-Kutta expression for the new time $t+h$ for small h

$$y(t+h) = y(t) + \frac{h}{6}(k_1 + 4k_2 + k_3) \qquad \text{with} \qquad \begin{cases} k_1 = f(t,\, y) \\ k_2 = f(t + \frac{h}{2},\, y + \frac{h}{2}\, k_1) \\ k_3 = f(t+h,\, y - h\, k_1 + 2\, h\, k_2) \end{cases}$$

is equivalent to the Taylor expansion

$$y(t+h) = y(t) + h\,\frac{dy}{dt} + \frac{h^2}{2}\,\frac{d^2y}{dt^2} + \frac{h^3}{6}\,\frac{d^3y}{dt^3} + \mathcal{O}(h^4)\,.$$

Use the Taylor expansion of a function g of two variables (u, v) around a given point (a, b)

$$g(u,v) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(u-a)^n\,(v-b)^m}{n!\,m!} \left(\frac{\partial^{n+m} g}{\partial u^n \partial v^m}\right)_{(u,v)=(a,b)}$$

and consider to use the simplified notation

$$y(t) \equiv y\,, \qquad f(t,y) \equiv f\,, \qquad \frac{\partial f}{\partial t} \equiv f_t \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv f_y\,.$$

For example,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t} \equiv f_t + f_y\, f\,.$$

Exercise 2 [*Pendulum motion*]

Consider the so-called *simple* pendulum (mass m and length l), for which the only external considered force is the gravitational one, $\vec{F} = -mg\vec{e}_z$. The angle ϕ describes the angular displacement of the pendulum from the neutral position $\phi = 0$. Suppose to have an initial displacement $\phi(t=0) = \phi_0$.

(i) Derive the equation of motion of the pendulum.

(ii) Solve analytically the equation of motion assuming $\phi_0 \ll 1$.

(iii) Implement the 4^{th} order Runge-Kutta method discussed in the lecture, in order to solve the equation of motion obtained in task (i). How can you solve a second order ordinary differential equation with Runge-Kutta methods? What would you do if you had to solve a higher order ordinary differential equation?

(iv) Test your code in the small-angle regime. Set ϕ_0 properly and compare the numerical solution to the previously calculated analytical one.

(v) Use your program to solve the equation of motion for initial conditions $\phi_0 \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$. Calculate the motion $\phi(t)$ for at least a complete period of the pendulum. Compare again your analytical solution with the numerical one. Can you explain the discrepancy?

(vi) Set $\phi = \frac{\pi}{2}$ and think of a way to calculate the period of the pendulum. Provide an error estimate for your result.