Numerische Methoden der Physik

WiSe 2023-2024 – Prof. Marc Wagner

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Exercise sheet 2

To be handed in on the 25.10.23 and discussed on 27.10.23 and 30.10.23

Exercise 1 [*Golden ratio*] `(3+2+5=10 pts.)`

(i) It is well known, that the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is the limit of the ratio of consecutive Fibonacci numbers $F(n-1)$ and $F(n)$. Write a short program to calculate

$$\delta(n) \equiv \frac{F(n)}{F(n-1)} - \varphi$$

and plot $\delta(n)$ as function of n. How does the plot change using `single` and `double` precision? What happens for large n?

(ii) Prove that, for any value of n,

$$\phi_\pm^{n+1} = \phi_\pm^{n-1} - \phi_\pm^{n} \ ,$$

where

$$\phi_\pm = \frac{-1 \pm \sqrt{5}}{2} \ .$$

(iii) Implement a short program to calculate the first 20 powers of $\phi_\pm$

(a) both using the iterative formula given above

(b) and raising $\phi_\pm$ directly to the given power.

Repeat both strategies in `single` and `double` precision. Can you explain what happens?

Exercise 2 [*Third order Runge-Kutta method*] `(10 pts.)`

Consider the differential equation $\dot{y}(t) = f(t, y)$, with f being at least 2-times differentiable. The Runge-Kutta method is a numerical procedure to iteratively obtain an approximate solution for $y(t)$. Show that for a given time t the third-order Runge-Kutta expression for the new time $t + \tau$ for small τ

$$y(t+\tau) = y(t) + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right) \qquad \text{with} \qquad \begin{cases} k_1 = f(t,\, y)\,\tau \\ k_2 = f\Big(t + \dfrac{\tau}{2},\, y + \dfrac{1}{2}\,k_1\Big)\,\tau \\ k_3 = f(t+\tau,\, y - k_1 + 2\,k_2)\,\tau \end{cases}$$

is equivalent to the Taylor expansion

$$y(t+\tau) = y(t) + \tau\,\frac{dy}{dt} + \frac{\tau^2}{2}\,\frac{d^2y}{dt^2} + \frac{\tau^3}{6}\,\frac{d^3y}{dt^3} + \mathcal{O}(\tau^4) \ .$$

Use the Taylor expansion of a function g of two variables (u, v) around a given point (a, b)

$$g(u, v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(u-a)^n \, (v-b)^m}{n! \, m!} \left(\frac{\partial^{n+m} g}{\partial u^n \partial v^m} \right)_{(u,v)=(a,b)}$$

and consider to use the simplified notation

$$y(t) \equiv y \, , \qquad f(t, y) \equiv f \, , \qquad \frac{\partial f}{\partial t} \equiv f_t \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv f_y \; .$$

For example,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \frac{dy}{dt} \equiv f_t + f_y \, f \; .$$