# Exercise sheet IX

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## Wick theorem

This exercise is a little bit tricky, because huge expressions arise: if you do not devise a good way to figure out the calculation, you will get lost very soon!

So, let us understand how to compute

$$\mathrm{T}\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\} \tag{1}$$

in the $\phi^4$ theory, *without* using the Wick theorem. First of all, we have to introduce some useful notation[1]. Writing $\phi_a$ instead of $\phi(x_a)$, we can split the field $\phi_a$ as following:

$$\phi_a = \phi_a^+ + \phi_a^- \;, \tag{2}$$

where

$$\phi_a^+ = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E(\mathbf{p})}}\,e^{-\imath p\cdot x_a}\,a(\mathbf{p}) \quad ; \tag{3a}$$

$$\phi_a^- = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E(\mathbf{p})}}\,e^{+\imath p\cdot x_a}\,a^\dagger(\mathbf{p}) \quad , \tag{3b}$$

so that the creation operator is separated from the annihilation one. Later, we will also need the concept of *normal order*; so let us define the normal ordering symbol $N()$ to place whatever operators in normal order, for example

$$N\big(a(\mathbf{p})\,a(\mathbf{k})^\dagger a(\mathbf{q})\big) \equiv a(\mathbf{k})^\dagger a(\mathbf{p})\,a(\mathbf{q}) \;. \tag{4}$$

The meaning of the "time–ordering" symbol T is well known: it instructs us to place the operators that follow in order with the latest to the left. Hence, the Eq. (1) could be rewritten explicitly:

$$\begin{aligned}
\mathrm{T}\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\} &= \theta(x_1^0-x_2^0)\,\theta(x_2^0-x_3^0)\,\theta(x_3^0-x_4^0)\,\phi_1\,\phi_2\,\phi_3\,\phi_4 + \\
&+ \theta(x_1^0-x_2^0)\,\theta(x_2^0-x_4^0)\,\theta(x_4^0-x_3^0)\,\phi_1\,\phi_2\,\phi_4\,\phi_3 + \theta(x_1^0-x_4^0)\,\theta(x_4^0-x_2^0)\,\theta(x_2^0-x_3^0)\,\phi_1\,\phi_4\,\phi_2\,\phi_3 + \\
&+ \theta(x_4^0-x_1^0)\,\theta(x_1^0-x_2^0)\,\theta(x_2^0-x_3^0)\,\phi_4\,\phi_1\,\phi_2\,\phi_3 + \theta(x_1^0-x_3^0)\,\theta(x_3^0-x_2^0)\,\theta(x_2^0-x_4^0)\,\phi_1\,\phi_3\,\phi_2\,\phi_4 + \\
&+ \theta(x_1^0-x_3^0)\,\theta(x_3^0-x_4^0)\,\theta(x_4^0-x_2^0)\,\phi_1\,\phi_3\,\phi_4\,\phi_2 + \theta(x_1^0-x_4^0)\,\theta(x_4^0-x_3^0)\,\theta(x_3^0-x_2^0)\,\phi_1\,\phi_4\,\phi_3\,\phi_2 + \\
&+ \theta(x_4^0-x_1^0)\,\theta(x_1^0-x_3^0)\,\theta(x_3^0-x_2^0)\,\phi_4\,\phi_1\,\phi_3\,\phi_2 + \theta(x_3^0-x_1^0)\,\theta(x_1^0-x_2^0)\,\theta(x_2^0-x_4^0)\,\phi_3\,\phi_1\,\phi_2\,\phi_4 + \\
&+ \theta(x_3^0-x_1^0)\,\theta(x_1^0-x_4^0)\,\theta(x_4^0-x_2^0)\,\phi_3\,\phi_1\,\phi_4\,\phi_2 + \theta(x_3^0-x_4^0)\,\theta(x_4^0-x_1^0)\,\theta(x_1^0-x_2^0)\,\phi_3\,\phi_4\,\phi_1\,\phi_2 + \\
&+ \theta(x_4^0-x_3^0)\,\theta(x_3^0-x_1^0)\,\theta(x_1^0-x_2^0)\,\phi_4\,\phi_3\,\phi_1\,\phi_2 + \theta(x_2^0-x_1^0)\,\theta(x_1^0-x_3^0)\,\theta(x_3^0-x_4^0)\,\phi_2\,\phi_1\,\phi_3\,\phi_4 + \\
&+ \theta(x_2^0-x_1^0)\,\theta(x_1^0-x_4^0)\,\theta(x_4^0-x_3^0)\,\phi_2\,\phi_1\,\phi_4\,\phi_3 + \theta(x_2^0-x_4^0)\,\theta(x_4^0-x_1^0)\,\theta(x_1^0-x_3^0)\,\phi_2\,\phi_4\,\phi_1\,\phi_3 + \\
&+ \theta(x_4^0-x_2^0)\,\theta(x_2^0-x_1^0)\,\theta(x_1^0-x_3^0)\,\phi_4\,\phi_2\,\phi_1\,\phi_3 + \theta(x_2^0-x_3^0)\,\theta(x_3^0-x_1^0)\,\theta(x_1^0-x_4^0)\,\phi_2\,\phi_3\,\phi_1\,\phi_4 + \\
&+ \theta(x_2^0-x_3^0)\,\theta(x_3^0-x_4^0)\,\theta(x_4^0-x_1^0)\,\phi_2\,\phi_3\,\phi_4\,\phi_1 + \theta(x_2^0-x_4^0)\,\theta(x_4^0-x_3^0)\,\theta(x_3^0-x_1^0)\,\phi_2\,\phi_4\,\phi_3\,\phi_1 + \\
&+ \theta(x_4^0-x_2^0)\,\theta(x_2^0-x_3^0)\,\theta(x_3^0-x_1^0)\,\phi_4\,\phi_2\,\phi_3\,\phi_1 + \theta(x_3^0-x_2^0)\,\theta(x_2^0-x_1^0)\,\theta(x_1^0-x_4^0)\,\phi_3\,\phi_2\,\phi_1\,\phi_4 + \\
&+ \theta(x_3^0-x_2^0)\,\theta(x_2^0-x_4^0)\,\theta(x_4^0-x_1^0)\,\phi_3\,\phi_2\,\phi_4\,\phi_1 + \theta(x_3^0-x_4^0)\,\theta(x_4^0-x_2^0)\,\theta(x_2^0-x_1^0)\,\phi_3\,\phi_4\,\phi_2\,\phi_1 + \\
&+ \theta(x_4^0-x_3^0)\,\theta(x_3^0-x_2^0)\,\theta(x_2^0-x_1^0)\,\phi_4\,\phi_3\,\phi_2\,\phi_1 \;.
\end{aligned} \tag{5}$$

Obviously, the idea of the exercise is to figure out one of these terms and not to calculate the whole expression by brute force! Without loss of generality, we can evaluate the term $\phi_i\,\phi_j\,\phi_k\,\phi_l$, where $i, j, k, l$ are indices running from 1 to 4.

[1] See M. Peskin & D. Schroeder, *An introduction to QFT*.

Using Eq. (2) and recalling[2] that $BA = AB + [A, B]$, we have:

$$
\begin{aligned}
&\phi_i\,\phi_j\,\phi_k\,\phi_l = \big(\phi_i^+ + \phi_i^-\big)\big(\phi_j^+ + \phi_j^-\big)\big(\phi_k^+ + \phi_k^-\big)\big(\phi_l^+ + \phi_l^-\big) = \\
&= \big(\phi_i^+\phi_j^+ + \phi_i^+\phi_j^- + \phi_i^-\phi_j^+ + \phi_i^-\phi_j^-\big)\big(\phi_k^+ + \phi_k^-\big)\big(\phi_l^+ + \phi_l^-\big) = \\
&= \Big(\phi_i^+\phi_j^+ + \phi_j^-\phi_i^+ + \phi_i^-\phi_j^+ + \phi_i^-\phi_j^- + \big[\phi_i^+, \phi_j^-\big]\Big)\big(\phi_k^+ + \phi_k^-\big)\big(\phi_l^+ + \phi_l^-\big) = \\
&= \Big(\phi_i^+\phi_j^+\phi_k^+ + \phi_i^-\phi_j^+\phi_k^+ + \phi_j^-\phi_i^+\phi_k^+ + \phi_i^-\phi_j^-\phi_k^+ + \big[\phi_i^+, \phi_j^-\big]\phi_k^+ + \\
&\quad + \phi_i^+\phi_j^+\phi_k^- + \phi_i^-\phi_j^+\phi_k^- + \phi_j^-\phi_i^+\phi_k^- + \phi_i^-\phi_j^-\phi_k^- + \big[\phi_i^+, \phi_j^-\big]\phi_k^-\Big)\big(\phi_l^+ + \phi_l^-\big) = \\
&= \Big(\phi_i^+\phi_j^+\phi_k^+ + \phi_i^-\phi_j^+\phi_k^+ + \phi_j^-\phi_i^+\phi_k^+ + \phi_i^-\phi_j^-\phi_k^+ + \big[\phi_i^+, \phi_j^-\big]\big(\phi_k^+ + \phi_k^-\big) + \phi_k^-\phi_i^+\phi_j^+ + \phi_i^+\big[\phi_j^+, \phi_k^-\big] + \\
&\quad + \big[\phi_i^+, \phi_k^-\big]\phi_j^+ + \phi_i^-\phi_k^-\phi_j^+ + \phi_i^-\big[\phi_j^+, \phi_k^-\big] + \phi_j^-\phi_k^-\phi_i^+ + \phi_j^-\big[\phi_i^+, \phi_k^-\big] + \phi_i^-\phi_j^-\phi_k^-\Big)\big(\phi_l^+ + \phi_l^-\big) = \\
&= \Big(\phi_i^+\phi_j^+\phi_k^+\phi_l^+ + \phi_i^-\phi_j^+\phi_k^+\phi_l^+ + \phi_j^-\phi_i^+\phi_k^+\phi_l^+ + \phi_i^-\phi_j^-\phi_k^+\phi_l^+ + \phi_k^-\phi_i^+\phi_j^+\phi_l^+ + \phi_i^-\phi_k^-\phi_j^+\phi_l^+ + \\
&\quad + \phi_j^-\phi_k^-\phi_i^+\phi_l^+ + \phi_i^-\phi_j^-\phi_k^-\phi_l^+ + \phi_i^+\phi_j^+\phi_k^+\phi_l^- + \phi_i^-\phi_j^+\phi_k^+\phi_l^- + \phi_j^-\phi_i^+\phi_k^+\phi_l^- + \phi_i^-\phi_j^-\phi_k^+\phi_l^- + \\
&\quad + \phi_k^-\phi_i^+\phi_j^+\phi_l^- + \phi_i^-\phi_k^-\phi_j^+\phi_l^- + \phi_j^-\phi_k^-\phi_i^+\phi_l^- + \phi_i^-\phi_j^-\phi_k^-\phi_l^- + \big[\phi_i^+, \phi_j^-\big]\big(\phi_k^+\phi_l^+ + \phi_k^-\phi_l^+ + \phi_k^+\phi_l^- + \phi_k^-\phi_l^-\big) + \\
&\quad + \big[\phi_j^+, \phi_k^-\big]\big(\phi_i^+\phi_l^+ + \phi_i^-\phi_l^+ + \phi_i^+\phi_l^- + \phi_i^-\phi_l^-\big) + \big[\phi_i^+, \phi_k^-\big]\big(\phi_j^+\phi_l^+ + \phi_j^-\phi_l^+ + \phi_j^+\phi_l^- + \phi_j^-\phi_l^-\big)\Big) = \\
&= \Big(\phi_i^+\phi_j^+\phi_k^+\phi_l^+ + \phi_i^-\phi_j^+\phi_k^+\phi_l^+ + \phi_j^-\phi_i^+\phi_k^+\phi_l^+ + \phi_i^-\phi_j^-\phi_k^+\phi_l^+ + \phi_k^-\phi_i^+\phi_j^+\phi_l^+ + \phi_i^-\phi_k^-\phi_j^+\phi_l^+ + \\
&\qquad + \phi_j^-\phi_k^-\phi_i^+\phi_l^+ + \phi_i^-\phi_j^-\phi_k^-\phi_l^+ + \phi_l^-\phi_i^+\phi_j^+\phi_k^+ + \phi_i^-\phi_l^-\phi_j^+\phi_k^+ + \phi_j^-\phi_l^-\phi_i^+\phi_k^+ + \\
&\qquad + \phi_i^-\phi_j^-\phi_l^-\phi_k^+ + \phi_k^-\phi_l^-\phi_i^+\phi_j^+ + \phi_i^-\phi_k^-\phi_l^-\phi_j^+ + \phi_j^-\phi_k^-\phi_l^-\phi_i^+ + \phi_i^-\phi_j^-\phi_k^-\phi_l^- + \\
&\qquad + \big[\phi_i^+, \phi_j^-\big]\big(\phi_k^+\phi_l^+ + \phi_k^-\phi_l^+ + \phi_l^-\phi_k^+ + \phi_k^-\phi_l^-\big) + \big[\phi_i^+, \phi_k^-\big]\big(\phi_j^+\phi_l^+ + \phi_j^-\phi_l^+ + \phi_l^-\phi_j^+ + \phi_j^-\phi_l^-\big) + \\
&\qquad + \big[\phi_i^+, \phi_l^-\big]\big(\phi_j^+\phi_k^+ + \phi_j^-\phi_k^+ + \phi_k^-\phi_j^+ + \phi_j^-\phi_k^-\big) + \big[\phi_j^+, \phi_k^-\big]\big(\phi_i^+\phi_l^+ + \phi_i^-\phi_l^+ + \phi_l^-\phi_i^+ + \phi_i^-\phi_l^-\big) + \\
&\qquad + \big[\phi_j^+, \phi_l^-\big]\big(\phi_i^+\phi_k^+ + \phi_i^-\phi_k^+ + \phi_k^-\phi_i^+ + \phi_i^-\phi_k^-\big) + \big[\phi_k^+, \phi_l^-\big]\big(\phi_i^+\phi_j^+ + \phi_i^-\phi_j^+ + \phi_j^-\phi_i^+ + \phi_i^-\phi_j^-\big) + \\
&\qquad + \big[\phi_i^+, \phi_j^-\big]\cdot\big[\phi_k^+, \phi_l^-\big] + \big[\phi_j^+, \phi_k^-\big]\cdot\big[\phi_i^+, \phi_l^-\big] + \big[\phi_i^+, \phi_k^-\big]\cdot\big[\phi_j^+, \phi_l^-\big]\Big) \\
&= N(\phi_i\,\phi_j\,\phi_k\,\phi_l) + \big[\phi_i^+, \phi_j^-\big]\cdot N(\phi_k\,\phi_l) + \big[\phi_i^+, \phi_k^-\big]\cdot N(\phi_j\,\phi_l) + \big[\phi_i^+, \phi_l^-\big]\cdot N(\phi_j\,\phi_k) + \\
&\qquad + \big[\phi_j^+, \phi_k^-\big]\cdot N(\phi_i\,\phi_l) + \big[\phi_j^+, \phi_l^-\big]\cdot N(\phi_i\,\phi_k) + \big[\phi_k^+, \phi_l^-\big]\cdot N(\phi_i\,\phi_j) \\
&\qquad + \big[\phi_i^+, \phi_j^-\big]\cdot\big[\phi_k^+, \phi_l^-\big] + \big[\phi_j^+, \phi_k^-\big]\cdot\big[\phi_i^+, \phi_l^-\big] + \big[\phi_i^+, \phi_k^-\big]\cdot\big[\phi_j^+, \phi_l^-\big]\,.
\end{aligned}
\tag{6}
$$

Some remarks should be done:

- it is better to normal order the products step by step, than to expand the whole product of four $\phi$ fields and normal ordering later the sixteen addends;
- in the last but one step some details of the calculation have been omitted, but they are completely similiar to previous ones;
- the commutator between two $\phi^\pm$ commutes with any other $\phi^\pm$;
- in the last step Eq. (4) was used.

The underlying idea of the calculation in Eq. (6) is to normal order all terms, because the expectation value of a normal ordered product of creation and annihilation operators vanishes. So, in Eq. (6), the only interesting

[2] Here, square brackets indicate the *commutator*.

terms to figure out are the last three products of commutators; we will calculate such commutators later. Now we should understand how to make Eq. (6) interact with Eq. (5). At first, one could put Eq. (6) inside Eq. (5), substituting $i, j, k, l$ indices with the right values, but this is neither amusing nor so bright. One possibility to reach our goal is to deduce what happens to the r.h.s. of Eq. (6) when two indices are switched; this is useful because to insert Eq. (6) in Eq. (5) means rearrange $i, j, k, l$ indices according to temporal coordinates order. As it is not so straightforward, let us proceed to separate Eq. (6) in three parts:

$$\phi_i\,\phi_j\,\phi_k\,\phi_l = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 \tag{7}$$

where

$$\mathcal{P}_1 \equiv N(\phi_i\,\phi_j\,\phi_k\,\phi_l) \tag{8a}$$

$$\begin{aligned}\mathcal{P}_2 \equiv{} & \left[\phi_i^+, \phi_j^-\right] \cdot N(\phi_k\,\phi_l) + \left[\phi_i^+, \phi_k^-\right] \cdot N(\phi_j\,\phi_l) + \left[\phi_i^+, \phi_l^-\right] \cdot N(\phi_j\,\phi_k) + \\ & + \left[\phi_j^+, \phi_k^-\right] \cdot N(\phi_i\,\phi_l) + \left[\phi_j^+, \phi_l^-\right] \cdot N(\phi_i\,\phi_k) + \left[\phi_k^+, \phi_l^-\right] \cdot N(\phi_i\,\phi_j)\end{aligned} \tag{8b}$$

$$\mathcal{P}_3 \equiv \left[\phi_i^+, \phi_j^-\right] \cdot \left[\phi_k^+, \phi_l^-\right] + \left[\phi_j^+, \phi_k^-\right] \cdot \left[\phi_i^+, \phi_l^-\right] + \left[\phi_i^+, \phi_k^-\right] \cdot \left[\phi_j^+, \phi_l^-\right]\,. \tag{8c}$$

It is obvious that $\mathcal{P}_1$ is not affected by any exchange of a couple of indeces, so for instance we have

$$N(\phi_i\,\phi_j\,\phi_k\,\phi_l) = N(\phi_j\,\phi_i\,\phi_k\,\phi_l)\,.$$

Hence, in Eq. (5) we will have $N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$ as a first term, not multiplied by any $\theta$–function.

Let us now consider $\mathcal{P}_2$. It is simple to notice that the fields inside commutators are all time–ordered, with the latest on the left. Thus, if we switch two indeces, we will obtain some commutators inverted; for example:

- $x_i > x_j > x_k > x_l$
$$\begin{aligned}\Rightarrow\quad \mathcal{P}_2 ={} & \left[\phi_i^+, \phi_j^-\right] \cdot N(\phi_k\,\phi_l) + \left[\phi_i^+, \phi_k^-\right] \cdot N(\phi_j\,\phi_l) + \left[\phi_i^+, \phi_l^-\right] \cdot N(\phi_j\,\phi_k) + \\ & + \left[\phi_j^+, \phi_k^-\right] \cdot N(\phi_i\,\phi_l) + \left[\phi_j^+, \phi_l^-\right] \cdot N(\phi_i\,\phi_k) + \left[\phi_k^+, \phi_l^-\right] \cdot N(\phi_i\,\phi_j)\,;\end{aligned}$$
- $x_j > x_i > x_k > x_l$
$$\begin{aligned}\Rightarrow\quad \mathcal{P}_2 ={} & \left[\phi_j^+, \phi_i^-\right] \cdot N(\phi_k\,\phi_l) + \left[\phi_j^+, \phi_k^-\right] \cdot N(\phi_i\,\phi_l) + \left[\phi_j^+, \phi_l^-\right] \cdot N(\phi_i\,\phi_k) + \\ & + \left[\phi_i^+, \phi_k^-\right] \cdot N(\phi_j\,\phi_l) + \left[\phi_i^+, \phi_l^-\right] \cdot N(\phi_j\,\phi_k) + \left[\phi_k^+, \phi_l^-\right] \cdot N(\phi_j\,\phi_i)\,;\end{aligned}$$
- $x_k > x_j > x_l > x_i$
$$\begin{aligned}\Rightarrow\quad \mathcal{P}_2 ={} & \left[\phi_k^+, \phi_j^-\right] \cdot N(\phi_l\,\phi_i) + \left[\phi_k^+, \phi_l^-\right] \cdot N(\phi_j\,\phi_i) + \left[\phi_k^+, \phi_i^-\right] \cdot N(\phi_j\,\phi_l) + \\ & + \left[\phi_j^+, \phi_l^-\right] \cdot N(\phi_k\,\phi_i) + \left[\phi_j^+, \phi_i^-\right] \cdot N(\phi_k\,\phi_l) + \left[\phi_l^+, \phi_i^-\right] \cdot N(\phi_k\,\phi_j)\,.\end{aligned}$$

Considering again that the ordering symbol $N()$ does not depend on the field order in his argument, it is not too hard to realize that in Eq. (5) we will have the following six terms:

$$\overbracket{\phi_1\phi_2}\,N(\phi_3\,\phi_4) + \overbracket{\phi_1\phi_3}\,N(\phi_2\,\phi_4) + \overbracket{\phi_1\phi_4}\,N(\phi_2\,\phi_3) + \overbracket{\phi_2\phi_3}\,N(\phi_1\,\phi_4) + \overbracket{\phi_2\phi_4}\,N(\phi_1\,\phi_3) + \overbracket{\phi_3\phi_4}\,N(\phi_1\,\phi_2)\,, \tag{9}$$

where we used the *contraction symbol* of two fields,

$$\overbracket{\phi_1\phi_2} \equiv \theta(x_1^0 - x_2^0)\,\left[\phi_1^+, \phi_2^-\right] + \theta(x_2^0 - x_1^0)\,\left[\phi_2^+, \phi_1^-\right]\,. \tag{10}$$

Finally, let us concentrate on $\mathcal{P}_3$. As for $\mathcal{P}_2$, also here all fields inside commutators are time–ordered; therefore we will have again some commutators inverted if we switch two indeces. For instance:

- $x_i > x_j > x_k > x_l$
$$\Rightarrow\quad \mathcal{P}_3 = \left[\phi_i^+, \phi_j^-\right] \cdot \left[\phi_k^+, \phi_l^-\right] + \left[\phi_j^+, \phi_k^-\right] \cdot \left[\phi_i^+, \phi_l^-\right] + \left[\phi_i^+, \phi_k^-\right] \cdot \left[\phi_j^+, \phi_l^-\right]\,;$$
- $x_l > x_k > x_j > x_i$
$$\Rightarrow\quad \mathcal{P}_3 = \left[\phi_l^+, \phi_k^-\right] \cdot \left[\phi_j^+, \phi_i^-\right] + \left[\phi_l^+, \phi_j^-\right] \cdot \left[\phi_k^+, \phi_i^-\right] + \left[\phi_l^+, \phi_i^-\right] \cdot \left[\phi_k^+, \phi_j^-\right]\,.$$

Using contraction symbols, it should be now straightforward to understand that in Eq. (5) we will have these three terms:

$$\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \ . \tag{11}$$

Putting Eq. (9) together with Eq. (11) and $\mathcal{P}_1$, we conclude that

$$\begin{aligned}
\mathrm{T}\big\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\big\} &= N(\phi_1\,\phi_2\,\phi_3\,\phi_4) + \wick{\c\phi_1\c\phi_2}\,N(\phi_3\,\phi_4) + \wick{\c\phi_1\c\phi_3}\,N(\phi_2\,\phi_4) + \\
&+ \wick{\c\phi_1\c\phi_4}\,N(\phi_2\,\phi_3) + \wick{\c\phi_2\c\phi_3}\,N(\phi_1\,\phi_4) + \wick{\c\phi_2\c\phi_4}\,N(\phi_1\,\phi_3) + \\
&+ \wick{\c\phi_3\c\phi_4}\,N(\phi_1\,\phi_2) + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \ ,
\end{aligned} \tag{12}$$

that is nothing but what we would have found applying *Wick theorem*.

It remains to figure out the commutator $\big[\phi^+(x), \phi^-(y)\big]$. It is better to calculate directly the contraction between $\phi(x)$ and $\phi(y)$:

$$\wick{\c\phi_1\c\phi_2} = \begin{cases} \big[\phi^+(x), \phi^-(y)\big] & \text{if} \quad x^0 > y^0 \\ \big[\phi^+(y), \phi^-(x)\big] & \text{if} \quad y^0 > x^0 \end{cases} \ . \tag{13}$$

Using Eq. (3) and recalling[3] that $\big[a(\mathbf{p}), a(\mathbf{q})^\dagger\big] = (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q})$, we have:

$$\begin{aligned}
\big[\phi^+(x), \phi^-(y)\big] &= \iint \frac{d\mathbf{p}}{(2\pi)^3}\,\frac{d\mathbf{q}}{(2\pi)^3}\,\frac{1}{\sqrt{2E(\mathbf{p})2E(\mathbf{q})}}\; e^{-\imath p\cdot x}\,e^{\imath q\cdot y}\Big[a(\mathbf{p}), a(\mathbf{q})^\dagger\Big] = \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})} e^{-\imath p\cdot(x-y)} \quad ;
\end{aligned} \tag{14a}$$

$$\begin{aligned}
\big[\phi^+(y), \phi^-(x)\big] &= \iint \frac{d\mathbf{p}}{(2\pi)^3}\,\frac{d\mathbf{q}}{(2\pi)^3}\,\frac{1}{\sqrt{2E(\mathbf{p})2E(\mathbf{q})}}\; e^{-\imath p\cdot y}\,e^{\imath q\cdot x}\Big[a(\mathbf{p}), a(\mathbf{q})^\dagger\Big] = \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})} e^{+\imath p\cdot(x-y)} \quad .
\end{aligned} \tag{14b}$$

Now, it is quite easy to recognize the *Feynman propagator* $D_F(x-y)$ in the previous expressions. In fact, one can demonstrate that

$$\begin{aligned}
D(x-y) &\equiv \langle 0|\,\phi(x)\phi(y)\,|0\rangle = \big[\phi^+(x), \phi^-(y)\big] \\
D(y-x) &\equiv \langle 0|\,\phi(y)\phi(x)\,|0\rangle = \big[\phi^+(y), \phi^-(x)\big]
\end{aligned}$$

and hence

$$\begin{aligned}
\wick{\c\phi_1\c\phi_2} &\equiv \theta(x_1^0 - x_2^0)\,\left[\phi_1^+, \phi_2^-\right] + \theta(x_2^0 - x_1^0)\,\left[\phi_2^+, \phi_1^-\right] \\
&= \theta(x_1^0 - x_2^0)\,\langle 0|\,\phi_1\phi_2\,|0\rangle + \theta(x_2^0 - x_1^0)\,\langle 0|\,\phi_2\phi_1\,|0\rangle = D_F(x-y) \ .
\end{aligned} \tag{15}$$

Finally, introducing Eq. (15) in Eq. (12), we get[4]:

$$\begin{aligned}
\mathrm{T}\big\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\big\} &= N(\phi_1\,\phi_2\,\phi_3\,\phi_4) + D_F(x_1-x_2)\,N(\phi_3\,\phi_4) + D_F(x_1-x_3)\,N(\phi_2\,\phi_4) + \\
&+ D_F(x_1-x_4)\,N(\phi_2\,\phi_3) + D_F(x_2-x_3)\,N(\phi_1\,\phi_4) + D_F(x_2-x_4)\,N(\phi_1\,\phi_3) + \\
&+ D_F(x_3-x_4)\,N(\phi_1\,\phi_2) + D_F(x_1-x_2)\cdot D_F(x_3-x_4) + \\
&+ D_F(x_1-x_3)\cdot D_F(x_2-x_4) + D_F(x_1-x_4)\cdot D_F(x_2-x_3) \ .
\end{aligned} \tag{16}$$

[3] Always following M. Peskin & D. Schroeder, *An introduction to QFT*.

[4] Observe that if we took the vacuum expectation value of Eq. (16), only the last three products of Feynman propagators would not vanish:

$$\langle 0|\,\mathrm{T}\big\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\big\}\,|0\rangle = D_F(x_1-x_2)\cdot D_F(x_3-x_4) + D_F(x_1-x_3)\cdot D_F(x_2-x_4) + D_F(x_1-x_4)\cdot D_F(x_2-x_3) \ .$$