Exercise sheet VI

November 23 [correction: November 30]

Problem 1 [*Conservation laws in free systems*]

(i) Determine the continuity equations and conserved currents corresponding to Lorentz transformations (boosts and rotations) for a free real scalar field.

(ii) Do the same as in (i) for a system of n non-interacting particles in classical mechanics, which is invariant under rotations and Galilean transformations. Interpret the results as well as the results obtained in (i).

Problem 2 [*Canonical quantization of Klein-Gordon: some operators*] Let us consider a theory with a real Klein-Gordon field, ϕ. For its canonical quantization we make use of the creation and annhilation operators, namely

$$a^\dagger(\mathbf{k}) \equiv \int d^3x \, \left(E(\mathbf{k})\phi(\mathbf{x}) - i\pi(\mathbf{x})\right) e^{+i\mathbf{kx}} \, ,$$

and

$$a(\mathbf{k}) \equiv \int d^3x \, \left(E(\mathbf{k})\phi(\mathbf{x}) + i\pi(\mathbf{x})\right) e^{-i\mathbf{kx}} \, .$$

Derive the following results (that were not proven in the theory lectures)

(i) $[a(\mathbf{k}_1), a(\mathbf{k}_2)] = [a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] = 0$.

(ii) Obtain the Hamiltonian, which is

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\mathbf{k})} E(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \, ,$$

(iii) and the momentum operator:

$$\mathbf{P} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\mathbf{k})} \mathbf{k} a^\dagger(\mathbf{k}) a(\mathbf{k}) \, .$$

(iv) Finally evaluate the following commutators

$$[H, a^\dagger(\mathbf{k})] = E(\mathbf{k}) a^\dagger(\mathbf{k}) \, ,$$

and

$$[H, a(\mathbf{k})] = -E(\mathbf{k}) a(\mathbf{k}) \, .$$