Exercise sheet VII

November 30 [correction: December 7]

Problem 1 [*Ladder operators*] Redo the canonical quantization procedure for a real scalar field, ϕ, using the following definition of the creation and annihilation operators

$$a(\mathbf{k}) \equiv \int \mathrm{d}^3x\, (\alpha(\mathbf{k})\phi(\mathbf{x}) + i\beta(\mathbf{k})\pi(\mathbf{x}))\, e^{-i\mathbf{k}\mathbf{x}}\,,$$
$$a^\dagger(\mathbf{k}) \equiv \int \mathrm{d}^3x\, (\alpha(\mathbf{k})\phi(\mathbf{x}) - i\beta(\mathbf{k})\pi(\mathbf{x}))\, e^{+i\mathbf{k}\mathbf{x}}\,.$$

(i) Express the Hamiltonian, H, in terms of the new a and $a^\dagger$. Obtain the values for α and β that yield

$$H = \int \frac{\mathrm{d}^3\mathrm{p}}{(2\pi)^3} E(\mathrm{p}) a^\dagger(\mathbf{p}) a(\mathbf{p})\ (+E_{\text{vacuum}})\,.$$

(ii) How does the field operator, ϕ, read in terms of a and $a^\dagger$? Is this expression invariant under Lorentz transformations?

Problem 2 [*Quantization inside a box*] Following again the canonical approach, quantize a real scalar field inside a cubic box with edge length L. Impose periodic boundary conditions. Proceed in the following steps:

(i) For $L \to \infty$ the three dimensional Fourier transform can be defined according to

$$\tilde{f}(\mathbf{k}) \equiv \frac{1}{(3\pi)^{3/2}} \int \mathrm{d}^3x f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}\,,$$
$$f(\mathbf{x}) \equiv \frac{1}{(3\pi)^{3/2}} \int \mathrm{d}^3k \tilde{f}(\mathbf{k}) e^{+i\mathbf{k}\mathbf{x}}\,.$$

What are the corresponding expressions for finite L?

(ii) Define suitable creation and annihilation operators using the result of (i).

(iii) Proceed with the canonical quantization. If $|\psi\rangle$ is an eigenstate of H such that

$$H|\psi\rangle = E_\psi|\psi\rangle\,, \quad N(\mathbf{k})|\psi\rangle = n_\psi(\mathbf{k})|\psi\rangle\,,$$

compute, for the case of finite L, the following:

(a) The commutator $[a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)]$.

(b) The Hamiltonian operator.

(c) The *number* operator, $N(\mathbf{k})$.

(d) The energy eigenvalues.

(e) $N(\mathbf{k})a^\dagger(\mathbf{p})\,|\psi\rangle$.

(f) $N(\mathbf{k})a(\mathbf{p})\,|\psi\rangle$.

(g) $N(\mathbf{k})\,|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_j\rangle$, where $|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_j\rangle \equiv a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\ldots a^\dagger(\mathbf{k}_j)\,|0\rangle$.