Exercise sheet VIII

December 7 [correction: December 14]

Problem 1 [*Casimir force*] In 1948 Hendrick Casimir published a paper where he reported about a very interesting effect. Between two perfectly parallel *uncharged* conducting plates there is a small attractive force. This force can be seen as a measure of a shift in the energy density of the vacuum. Although the total energy density of the vacuum is not observable, Casimir realized that a shift on it has phyical consequences. The idea is that the vacuum energy between those plates is coming only from modes, whose wavelengths are a multiple of the distance between the plates (only such modes are compatible with the boundary conditions imposed by conducting plates). Therefore, the energy density between the plates must decrease as they come together, which means that there is an attractive force between the plates that increases when the gap decreases.

Let us compute the so-called *Casimir force*. In order to make things simpler we can consider a scalar field in a two dimensional spcace-time (one spatial dimenion plus time). The Lagrangian density is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi(x,t))(\partial^\mu \phi(x,t)) \, ,$$

and the equation of motion of the field ϕ reads

$$\Box \phi(x,t) = 0 \, .$$

(i) Find the solution of the equation of motion with "two metal plates", one at $x = 0$ and the other at $x = D$. Write down also the conjugate momentum $\pi(x,t)$.

(ii) Obtain the Hamiltonian in terms of the creation and annihilation operators. What is the vacuum energy in this case?

(iii) Compute the force between the two plates. We suggest the following path:

(a) Introduce an *ultraviolet cut-off*, λ, such that each energy contribution is multiplied by $e^{-\lambda k_n}$, with k_n being the contribution of the n-th mode to the vacuum energy density. At the end this cut-off can be removed by taking the limit $\lambda \to 0$.

(b) Put the two plates inside another two plates separated by a distance $L \gg D$. Write the energy density in that case, let us call it $\tilde{E}(\lambda, D, L)$.

(c) Determine the Casimir force as the derivative of the vacuum energy density with respect to the separation of the plates D, namely

$$F(D) = - \lim_{L \to \infty} \frac{\partial \tilde{E}(\lambda, D, L)}{\partial D} \, .$$