Exercise sheet X

`January 18 [correction: January 25]`

Problem 1 [*Unitary and hermitian matrices*] Show the following relationships between unitary and hermitian matrices:

(i) Any $n \times n$ unitary matrix U, i.e. $U^\dagger U = \mathbb{1}$, can be written as

$$U = \exp(iH)$$

where H is hermitian, $H = H^\dagger$.

(ii) $\det(U) = 1$ implies that H is traceless.

Note: This result means that $n \times n$ unitary matrices with unit determinant can be generated by $n \times n$ traceless hermitian matrices.

Problem 2 [*An identity for SU(2) matrices*] Prove the following identity for 2×2 unitary matrices generated by Pauli matrices:

$$\exp(i\mathbf{r} \cdot \sigma) = \cos r + (\hat{\mathbf{r}} \cdot \sigma) \sin r \,, \quad (1)$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$; $r = |\mathbf{r}|$ is the magnitude of the vector $\mathbf{r}$; and $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector.

Problem 3 [*Gammatics*] Using only the anticommutation relations of the gamma matrices, prove the following relations in a 4-dimensional space-time:

(i) $\mathrm{tr}(\gamma^{\mu_1} \ldots \gamma^{\mu_n}) = 0$, with n odd

(ii) $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

(iii) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$

(iv) $\mathrm{tr}(\gamma^5) = 0$

(v) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) = 0$

(vi) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$

(vii) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \ldots) = \mathrm{tr}(\ldots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu)$

(viii) $\gamma^\mu \gamma_\mu = 4$

(ix) $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

(x) $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}$

(xi) $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$

(xii) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24$

(xiii) $\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6\delta^\mu{}_\nu$

(xiv) $\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho)$