Exercise sheet I

October 15 [solution: October 24]

Problem 1 [*Natural units*] How much is $1\,kg$ in GeV and $1\,s$ in GeV^{-1}? Use the results to express Newton's gravitational constant $G_N = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and the value of the Planck mass $M_{\mathrm{Pl}} = G_N^{-1/2}$ in natural units.

Problem 2 [*Hamiltonian mechanics*] Starting from the definition of the Hamiltonian,

$$H(x,p) \equiv p\dot{x} - L(x,\dot{x}),$$

and using the Euler-Lagrange equation, derive Hamilton's equations,

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}$$

[**Hint:** Be careful about what the independent variables of a function are.]

Problem 3 [*Continuity equation*] Using the Schrödinger equation for the wavefunction $\Psi(\mathbf{x},t)$,

$$\left[-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x})\right]\Psi(\mathbf{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t),$$

show that the probability density, $\rho = \Psi^*\Psi$, satisfies the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla\mathbf{j} = 0$$

where

$$\mathbf{j} = \frac{\hbar}{2mi}\left[\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right].$$

Problem 4 [*Heisenberg equation of motion*] Let $\hat{O}$ be a time-independent operator in the Schrödinger picture, and $\hat{H}$ the time-independent Hamiltonian of the system. Starting from the definition of a Heisenberg operator,

$$\hat{O}_H(t) = e^{i\hat{H}(t-t_0)/\hbar}\hat{O}e^{-i\hat{H}(t-t_0)/\hbar},$$

derive the Heisenberg equation of motion,

$$i\hbar\frac{d\hat{O}_H}{dt} = [\hat{O}_H, \hat{H}_H].$$