Exercise sheet V

November 12 [solution: November 21]

Problem 1 [*Classical real scalar field*]

(i) Using the Euler-Lagrange equation, find the equation of motion of the real scalar (Klein-Gordon) field whose Lagrangian density is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{1}$$

(ii) Check that the solution of the Klein-Gordon equation is a plane wave:

$$\phi(x) = e^{\pm i p^\mu x_\mu} \equiv e^{\pm ipx}. \tag{2}$$

Justify that it implies that a general solution of this equation is:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E(\vec{p})}} \left(a(\vec{p})e^{-ipx} + a^*(\vec{p})e^{ipx} \right), \tag{3}$$

where $a(\vec{p})$ is an arbitrary function and $E(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$.

(iii) Calculate the energy-momentum tensor for the real scalar field:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \mathcal{L} g^{\mu\nu}, \tag{4}$$

where $g^{\mu\nu}$ is the metric tensor.

(iv) Using result from (iii) and eq. (3), calculate the total energy and momentum of the Klein-Gordon field:

$$H = \int T^{00} d^3x, \qquad P^i = \int T^{0i} d^3x. \tag{5}$$

Hint: use the complex representation of the 3D Dirac delta function:

$$\delta(\vec{p} \pm \vec{p}') = \frac{1}{(2\pi)^3} \int d^3x e^{i(\vec{p} \pm \vec{p}')\cdot\vec{x}}. \tag{6}$$

Problem 2 [*Classical complex scalar field*]

(i) Show that the complex Klein-Gordon field Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{7}$$

is invariant with respect to the global gauge transformation:

$$\phi \to \phi' = e^{-i\alpha}\phi, \qquad \phi^* \to (\phi^*)' = e^{i\alpha}\phi^*. \tag{8}$$

(ii) Calculate the conserved Noether current j^μ:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial \alpha}\Big|_{\alpha=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\partial \phi^*}{\partial \alpha}\Big|_{\alpha=0}, \tag{9}$$

related to the global gauge transformation (8). Check that $\partial_\mu j^\mu = 0$.