Exercise sheet VI

November 19 [solution: November 28]

Problem 1 [*Gauge invariance and mass*] Consider the two Lagrangians:

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{1}$$

$$\mathcal{L}_{Proca} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + m^2 A_\mu A^\mu. \tag{2}$$

$\mathcal{L}_{Maxwell}$ describes a massless spin-1 field, while $\mathcal{L}_{Proca}$ a massive spin-1 field.

(i) Derive equations of motion for both Lagrangians.

(ii) Show that $\mathcal{L}_{Maxwell}$ is and $\mathcal{L}_{Proca}$ is not invariant under the following gauge transformation:

$$A^\mu \to A^\mu + \partial^\mu \chi, \tag{3}$$

where χ is an arbitrary scalar function. Comment on the role of the mass term in $\mathcal{L}_{Proca}$.

(iii) Show that the components of the Proca equation (equation of motion resulting from the Proca Lagrangian) satisfy the Klein-Gordon equation with an additional constraint:

$$\partial_\mu A^\mu = 0. \tag{4}$$

How many physical degrees of freedom are described by A_μ? Compare to the number of physical degrees of freedom of A_μ for the massless case.

Problem 2 [*Quantum complex scalar field – particles and antiparticles*] Consider the quantum complex scalar field. Start with the conserved current $j^\mu = -i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi)$ and the conserved charge $Q = \int d^3x j^0$. Substitute the general solution for the field ϕ:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left(a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{+ikx} \right) \tag{5}$$

and show that:

$$Q = \int d^3k \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) - b^\dagger(\mathbf{k}) b^\dagger(\mathbf{k}) \right), \tag{6}$$

i.e. that the number operators $a^\dagger a$ and $b^\dagger b$ count the charges with **opposite** sign.

Problem 3 [*Quantum causality*] Using the general solution for ϕ (eq. (5)), show the quantum causality condition:

$$\left[\phi(x), \phi^\dagger(y) \right] = 0 \tag{7}$$

for space-like intervals $(x-y)^2 < 0$. This condition implies that fields separated by a space-like interval are independent (i.e. quantum complex scalar field is causal).