Exercise sheet VII

November 26 [solution: December 5]

Problem 1 [*Vacuum energy*] Use the results of Problem 1 of Exercise Sheet V and the commutation relations for the operators $a(\vec{k})$ and $a^\dagger(\vec{k}')$:

$$\Big[a(\vec{k}), a^\dagger(\vec{k}')\Big] = \delta^{(3)}(\vec{k} - \vec{k}') \tag{1}$$

to derive the total energy of the quantum real scalar field. Comment on the infinite term that you obtain, the *vacuum energy*.

Note: in the lecture a slightly different convention was used for the expansion of $\phi(x)$ and thus for the commutation relations of $a(\vec{k})$ and $a^\dagger(\vec{k}')$. To be consistent with our Eq. (3) of Exercise Sheet V, the commutation relations have to be as above.

Problem 2 [*Gauge and chiral transformations*] Show that the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) \tag{2}$$

is invariant with respect to the global gauge transformation:

$$\psi(x) \to e^{-i\alpha}\psi(x) \tag{3}$$

and, if $m = 0$, with respect to the chiral transformation:

$$\psi(x) \to e^{-i\alpha\gamma^5}\psi(x), \tag{4}$$

where:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{5}$$

Find the Noether currents related to these transformations. Check whether they are conserved.

Problem 3 [*Energy of the Dirac field*]

(i) Find the energy-momentum tensor for the Dirac field.

(ii) Calculate the total energy of the Dirac field:

$$H = \int d^3x\, T^{00} \tag{6}$$

and show that it can be written as:

$$H = \int d^3x\, \psi^\dagger H_D \psi, \tag{7}$$

where $H_D = -i\vec{alpha} \cdot \vec{\nabla} + \beta m$ is the so-called Dirac Hamiltonian ($\vec{\alpha} \equiv \gamma^0\vec{\gamma}$, $\beta \equiv \gamma^0$).