Exercise sheet X

December 17 [solution: January 16]

Problem 1 [*Feynman propagator for scalar field*] Consider the inhomogeneous Klein-Gordon equation:

$$(\partial^2 + m^2)\phi(x) = j(x), \tag{1}$$

which describes a scalar field $\phi(x)$ interacting with an external source $j(x)$. The solution of this equation can formally be written as:

$$\phi(x) = \phi_0(x) - \int d^4y G(x-y) j(y), \tag{2}$$

where $\phi_0(x)$ is the solution of the homogeneous Klein-Gordon equation and $G(x-y)$ is the Green's function satisfying:

$$(\partial^2 + m^2) G(x-y) = -\delta^{(4)}(x-y). \tag{3}$$

a) Write the equation for an inverse 4-dimensional Fourier transform of $G(x-y)$. Solve this equation in Fourier space (using Eq. (3)) and show that the Fourier transform of $G(x-y)$ is $G(k) = \frac{1}{k^2-m^2}$.
b) The obtained expression:

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2-m^2} \tag{4}$$

is, however, not well-defined – it has a singularity at $k^2 = m^2$, i.e. at $k^0 = \pm E(\mathbf{k})$. One way to regularize the integral is by using the Feynman prescription: the pole at $k^0 = \pm E(\mathbf{k})$ is moved to $k^0 = \pm E(\mathbf{k}) \mp i\epsilon$. Show that this leads to the Feynman propagator in momentum space:

$$\Delta_F(k) = \frac{1}{k^2 - m^2 + i\epsilon}. \tag{5}$$

c) Consider now the following integral that will give you the Feynman propagator in position space:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2-m^2+i\epsilon}. \tag{6}$$

Carry out the integral over k^0 **only**, using the residue theorem. You should finally obtain:

$$\Delta_F(x-y) = -i \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left(\theta(x^0 - y^0) e^{-ik(x-y)} + \theta(y^0 - x^0) e^{ik(x-y)} \right). \tag{7}$$

Try to interpret this equation.
Note: The Feynman propagator $\Delta_F(x-y)$ or $\Delta_F(k)$ is sometimes defined with an

additional factor of i, i.e. $\Delta_F(k) = \frac{i}{k^2-m^2+i\epsilon}$ and $\Delta_F(x-y)$ without the prefactor $(-i)$ in front of the integral in Eq. (7).

d) Show that the Feynman propagator in position space is the vacuum expectation value of the time-ordered product of 2 fields:

$$T\phi(x)\phi^\dagger(y) = \begin{cases} \phi(x)\phi^\dagger(y) & \text{for} \quad x^0 > y^0 \\ \phi^\dagger(y)\phi(x) & \text{for} \quad x^0 < y^0, \end{cases} \tag{8}$$

i.e.:

$$\langle 0|T\phi(x)\phi^\dagger(y)|0\rangle = i\Delta_F(x-y). \tag{9}$$

Hint: Insert explicitly:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{+ikx}\right) \tag{10}$$

and use the commutation relations:

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = (2\pi)^3 2E(\mathbf{k})\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{11}$$

and the action of the annihilation operators on the vacuum state.

Note: For the real scalar field, we have: $\langle 0|T\phi(x)\phi(y)|0\rangle = i\Delta_F(x-y)$.

Comment: Equation (9) is sometimes used as the **definition** of the Feynman propagator (sometimes without the i factor, which leads to the additional factor of i in Δ_F, see the note on the first page). From this definition, one can then go in the reverse direction than we did and arrive at the interpretation of the Feynman propagator as the Green's function of some differential equation, i.e. the Klein-Gordon equation.