Exercise sheet 0 (special session)

April 20, 2012

A short primer in complex analysis

(i) *Cauchy-Riemann conditions for a function* $f : E \to \mathbb{C}$, with $E \subseteq \mathbb{C}$: assuming Ref, Imf are differentiable (in the ordinary $\mathbb{R}$ sense), the conditions are:

$$\begin{cases} \frac{\partial \mathrm{Re} f}{\partial \mathrm{Re} z} & = & +\frac{\partial \mathrm{Im} f}{\partial \mathrm{Im} z} \; ; \\ \\ \frac{\partial \mathrm{Re} f}{\partial \mathrm{Im} z} & = & -\frac{\partial \mathrm{Im} f}{\partial \mathrm{Re} z} \; . \end{cases}$$

If C-R are satisfied, f is *differentiable in* z (in the complex sense); that is, regardless of the exact path for $h \to 0, h \in \mathbb{C}$, the derivative

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is *unique*.

"Differentiable" in practice means: no z^*, no $|z|$, no Rez or Imz explicitly; smooth functions of z only.

The derivative can be written in various forms using C-R: for instance,

$$f'(z) = \frac{\partial \mathrm{Re} f}{\partial \mathrm{Re} z} - i \frac{\partial \mathrm{Re} f}{\partial \mathrm{Im} z} \quad .$$

Def. f is *analytic* in $E \Leftrightarrow f$ is differentiable in $\forall\ z \in E$.

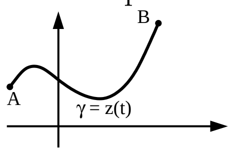
Def. f is *analytic* in $z_0 \Leftrightarrow$ there is a $I_\delta(z_0) = \{z \ \ / \ \ |z - z_0| < \delta\}$ where f is analytic.

(ii) *Integrals in the complex plane* $\mathbb{C}$.

- A *Jordan curve* $z(t) \in \mathbb{C}, t \in [a, b]$ is:
  (a) regular: z, $\frac{\mathrm{d}z}{\mathrm{d}t}$ continuous for $\forall\ t$;
  (b) closed: $z(a) = z(b)$;
  (c) simple: $\forall\ t_1, t_2 \in [a, b) : z(t_1) \neq z(t_2)$ (i.e. does not cross itself).

  Conventionally, its positive orientation is the counterclockwise one.
- A domain D is *simply connected* if each closed curve in D can be continuously deformed ("homotopically equivalent") to a point. In practice: "no holes".

- The complex integral on a path is given by:



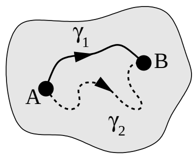
$$\begin{aligned} z(a) &= A \\ z(b) &= B \end{aligned}$$

$$\begin{aligned} \int_{A;\gamma}^{B} f(z)\mathrm{d}z &= \int_{z}^{b} f(z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t = (\cdots) = \\ &= \int_{A;\gamma}^{B} \Big[\mathrm{Re}f\mathrm{d}(\mathrm{Re}z) - \mathrm{Im}f\mathrm{d}(\mathrm{Im}z)\Big] + i\int_{A;\gamma}^{B} \Big[\mathrm{Re}f\mathrm{d}(\mathrm{Im}z) + \mathrm{Im}f\mathrm{d}(\mathrm{Re}z)\Big] \ . \end{aligned}$$

(iii) *Cauchy theorem*

- $f(z)$ analytic in E, E simply connected, γ Jordan curve in E: then,

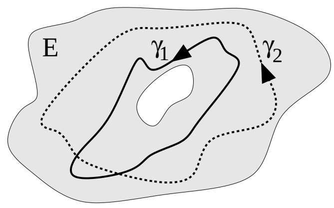
$$\int_{\gamma} f(z)\mathrm{d}z = 0 \ .$$

- *(corollary)* For two curves in E such as  (no holes inside!):

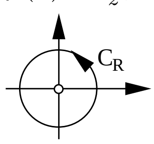
$$\int_{A;\gamma_1}^{B} f(z)\mathrm{d}z = \int_{A;\gamma_2}^{B} f(z)\mathrm{d}z \ .$$

- *(generalised form)* $f(z)$ analytic in E (which need not be simply connected anymore); γ_1, γ_2 Jordan curves *homotopically equivalent* in E (that is, deformed one onto another within E and continuously). Then

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z \ .$$



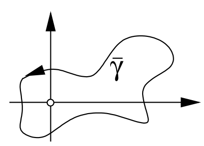
Example 1 $f(z) = \frac{1}{z}$, defined in $\mathbb{C}/\{0\}$ (one hole at the origin).



Curve C_R: $$\begin{aligned} z(\theta) &= Re^{i\theta} \ , \quad \theta \in [0, 2\pi) \ ; \\ \mathrm{d}z &= iRe^{i\theta} \ , \end{aligned}$$

$$I = \int_{C_R} f(z)\mathrm{d}z = \int_0^{2\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\mathrm{d}\theta = \int_0^{2\pi} i\mathrm{d}\theta = 2\pi i \neq 0 \ ,$$

nonzero, indeed there is a "hole" enclosed in C_R.

- Then, for any curve $\overline{\gamma}$ surrounding the origin, $\int_{\overline{\gamma}} f(z)\mathrm{d}z = 2\pi i$: 

- Also, one can prove $\int_{C_R} \frac{\mathrm{d}z}{z^n} = 0$, for any integer $n > 1$.

(iv) *Cauchy integral representation* (a.k.a. "for an analytic function, all derivatives exist"): If $f(z)$ analytic in E, E simply connected, and $\gamma = \partial S$ a Jordan curve which encloses the domain S, then:

$$\forall z \in S : f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(z')}{z' - z} \mathrm{d}z' \ .$$

(Corollary) for the n-th derivative, one differentiates inside the integral, getting:

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n} = \frac{n!}{2\pi i} \int_\gamma \frac{\mathrm{d}z' f(z')}{(z' - z)^{n+1}} \ .$$

Def. *Zeros of a function*

- f, analytic, is *regular in* z_0 if it is there defined.
- z_0 is a *zero of* f if $f(z_0) = 0$.
- One speaks of "zero of n-th order" if:

$$\begin{aligned} f(z_0) &= f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0 \ ; \\ f^{(n)}(z_0) &\neq 0 \ , \end{aligned}$$

  which is equivalent to

$$f(z) = (z - z_0)^n g(z) \ , \ \text{with} \begin{cases} g(z_0) \neq 0 \ , \\ g \text{ analytic in } z_0 \ . \end{cases}$$

Def. z_0 is an *isolated singularity* of f if there is a "holed domain" $\overline{I}(z_0) = I(z_0)/\{z_0\}$ such that $f(z)$ is regular in $\forall z \in \overline{I}(z_0)$.
(One cannot use a power series here; but something else can be done ...)

(v) *Laurent theorem, Laurent series.* If z_0 is an isolated singularity for $f(z)$, then there is a "holed domain" $\overline{I}(z_0)$ where

$$f(z) = \sum_{k=-\infty}^{+\infty} d_k (z - z_0)^k \ , \quad \forall z \in \overline{I}(z_0) \ ,$$

with coefficients given by

$$d_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z$$

(C is some circle centred in z_0 and contained in $\overline{I}(z_0)$).
Remark. The Taylor series is a particular case, with $d_k = 0$, $\forall k < 0$.

Def. *Poles, singularities*

- If $d_{-n} \neq 0$ and $d_{-n-1} = d_{-n-2} = \ldots = 0$, then z_0 is a *pole of order* n of $f(z)$; in that case (within $\overline{I}(z_0)$),

$$f(z) = \frac{1}{(z-z_0)^n} \sum_{k'=0}^{+\infty} d_{k'-n}(z-z_0)^{k'} = \frac{g(z)}{(z-z_0)^n} \quad , \quad \text{with} \begin{cases} g \text{ analytic in } z_0 \ , \\ g(z_0) \neq 0 \ , \end{cases}$$

(in practice, for a simple ($n=1$) pole, the function is locally $\sim \frac{1}{z-z_0}$).

- If, instead, there are *infinitely many* nonzero d_{negative}, z_0 is an *essential singularity* (e.g. $z_0 = 0$ for $f(z) = e^{1/z}$); f is then "a monster".

(vi) *Residues*: suppose z_0 is an isolated singularity ($\rightarrow f$ is regular in $\overline{I}(z_0)\ldots$).

Def. The *residue* of f at z_0 is defined as:

$$\mathrm{Res} f(z)|_{z_0} = \frac{1}{2\pi i} \int_C f(z) \mathrm{d}z \quad , \quad \text{with } C \in \overline{I}(z_0) \text{ surrounding } z_0 \ ,$$

from which it follows that $\mathrm{Res} f(z)|_{z_0} = d_{-1}$.

Note: z_0 regular implies zero residue, but zero residue does not imply anything! See, e.g., $f(z) = \frac{1}{z^2}$ at $z_0 = 0$.

Calculation (using the Cauchy representation)

– For a simple pole:

$$\mathrm{Res} f(z)|_{z_0} = \lim_{z \to z_0} (z - z_0) f(z) \ .$$

– For any n:

$$\mathrm{Res} f(z)|_{z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} [(z-z_0)^n f(z)] \ .$$

– The above are not valid for an essential singularity: there, the residue is read off the Laurent expansion. Example:

$$\begin{aligned} e^{1/z} &= 1 + \underbrace{\frac{1}{z}}_{(\star)} + \frac{1}{2!}\frac{1}{z^2} + \cdots \\ (\star) &= d_{-1}(z-0)^{-1} \Rightarrow \mathrm{Res}(e^{1/z})|_0 = 1 \ . \end{aligned}$$

(vii) *Residue theorem.* Suppose D is a compact domain (practically: closed and limited); the Jordan curve γ in D does not run over any singularity and encloses, on the inside, at most a *finite* number of isolated singularities $z_k, k = 1, \ldots, n$; $f(z)$ is regular in D – except at the singularities. Then

$$\int_\gamma f(z) \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res} f(z)|_{z_k} \ .$$

Applications. Two of the most typical classes of integrals that can be solved with these tools are:

(a) *Trigonometric integrals.* Suppose we must find

$$I_1 = \int_0^{2\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta \ ;$$

with the identities

$$\begin{aligned} \cos\theta &= \frac{e^{+i\theta} + e^{-i\theta}}{2} \ ; \\ \sin\theta &= \frac{e^{+i\theta} - e^{-i\theta}}{2i} \ , \end{aligned}$$

it can be made into a complex integral along the unit circle C: setting $z = e^{i\theta}$, we have:

$$\begin{aligned} \mathrm{d}z &= iz\mathrm{d}\theta \ ; \\ \cos\theta &= \frac{z + 1/z}{2} \ , \text{ and so on.} \end{aligned}$$

[*Question:* why $1/z$ instead of z^*? On the unit circle they are the same...]
The result is then ready to be solved with the residue theorem:

$$I_1 = \int_C \frac{f(z)}{iz}\mathrm{d}z \ .$$

(b) *Integrals on* $\mathbb{R}$.

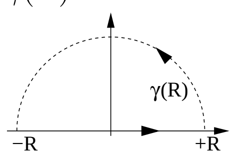
- An integral in the form

$$I_2 = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x$$

means, formally written (the last expression holds if the previous one is defined):

$$I_2 = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{+R_2} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{+R} f(x)\mathrm{d}x \ .$$

This can be embedded in $\mathbb{C}$, and a half-circle $\gamma(R)$ can be added with centre 0 and radius R, to make a Jordan curve:



It is now easy to use the residue theorem on

$$J(R) = \int_{-R}^{+R} f(x)\mathrm{d}x + \int_{\gamma(R)} f(z)\mathrm{d}z = 2\pi i \sum_{\dots} \mathrm{Res} f(z)|_{\dots} \ .$$

Under suitable conditions,

$$\lim_{R\to\infty} \int_{\gamma(R)} f(z)\mathrm{d}z = 0 \ ,$$

hence

$$I_2 = \lim_{R\to\infty} J(R) \ .$$

- The condition for this to hold (sometimes called mistakenly *Jordan lemma*) is:
$$f(z) = \mathrm{o}(1/z) \quad , \; |z| \to \infty \quad ;$$
indeed, with $z = Re^{i\theta}$:
$$\int_{\gamma(R)} f(z)\mathrm{d}z = iR\int_0^{\pi} \mathrm{d}\theta f(Re^{i\theta})e^{i\theta} \quad ,$$
taking the modulus,
$$\Big|\int_{\gamma(R)} f(z)\mathrm{d}z\Big| = \Big|iR\int_0^{\pi} \mathrm{d}\theta f(Re^{i\theta})e^{i\theta}\Big| \leq R\int_0^{\theta} |f(Re^{i\theta})| \quad ;$$
but
$$\lim_{R\to\infty}\Big[\underbrace{R}_{\sim(1/z)^{-1}}\underbrace{f(Re^{i\theta})}_{\mathrm{o}(1/z)}\Big] = 0 \quad ,$$
then
$$\Big|\int_{\gamma(R)} f(z)\mathrm{d}z\Big| \to 0 \quad , \quad \text{for } R \to \infty \quad .$$
- *Jordan lemma*. Assuming that $\alpha > 0$, $f(z) = \mathrm{o}(1)$ for $0 \leq \arg(z) \leq \pi$, and a half-circle $\gamma(R)$ as in the above picture, it is:
$$\lim_{R\to\infty}\int_{\gamma(R)} e^{i\alpha z} f(z)\mathrm{d}z = 0 \quad .$$

Note Similar consideration obviously hold, *mutatis mutandis*, if we are to close the path in the lower half-plane: in that case, besides the need for $\alpha < 0$, one has to include an additional minus sign from reversing the orientation of the curve. Also integrations on the imaginary axis can be treated with these devices, in that case with no i in the exponent.

Problem 1 [*Functional differentiation*] : The formal definition of the functional derivative is as follows: with $F[\phi]$ a functional of $\phi(x)$,

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\mathrm{d}}{\mathrm{d}s} F[\phi(x) + s\delta(x-y)]\Big|_{s=0} \ .$$

(i) In calculating the transition probability $\langle x_2, t_2 | x_1, t_1\rangle$ for the harmonic oscillator,

$$S[x] = \int_{t_1}^{t_2} \mathrm{d}u \left(\frac{m}{2}\dot{x}^2(u) - \frac{m\omega^2}{2} x^2(u) \right) \ ; \ \ x = x(t) \ ,$$

one can write the trajectory as $x(t) = x_{\mathrm{cl}}(t) + y(t)$, with boundary conditions $x_{\mathrm{cl}}(t_1) = x_1$, $x_{\mathrm{cl}}(t_2) = x_2$ and $y(t_1) = y(t_2) = 0$.

Expanding around the classical solution to the equations of motion, x_{cl}, the following exact result holds:

$$S[x] = S[x_{\mathrm{cl}}] + \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{m}{2}\dot{y}^2(t) - \frac{m\omega^2}{2} y^2(t) \right) \ .$$

(a) Show, using the formal definition above, that indeed

$$\frac{\delta S[x]}{\delta x(t)} = -m\ddot{x}(t) - m\omega^2 x(t) \ .$$

(b) Show that

$$\frac{\delta^2 S[x]}{\delta x(t)\delta x(t')} = \delta(t-t')\Big(-m\frac{\mathrm{d}^2}{\mathrm{d}t'^2} - m\omega^2 \Big) \ .$$

(c) Show that, for any $n > 2$,

$$\frac{\delta^n S[x]}{\delta x(t)\delta x(t')\cdots\delta x(t^{(n-1)})} = 0 \ ;$$

for which class of potentials do you expect this result to hold?

(ii) Starting from the real Klein-Gordon action in four dimensions, with metric $g^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$,

$$S[\phi] = \int \mathrm{d}^4x \Big(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi^2(x) \Big) \ ,$$

evaluate the derivative

$$\frac{\delta S[\phi]}{\delta\phi(x)}$$

and verify that setting it to zero yields the Klein-Gordon equation $(\Box - m^2)\phi(x) = 0$. [*Hint*: in integration by parts, the finite contribution can be neglected.]

Problem 2 [*Contour integration, residues*] :

(i) Prove that

$$I_1 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 3\cos\theta} = \frac{\pi}{2} \ .$$

(ii) Prove that

$$I_2 = \int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \mathrm{d}x = \frac{\pi}{6} \ .$$

(iii) Prove that

$$I_3 = \int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2} \mathrm{d}x = \frac{\pi}{e} \ .$$

Hint: different parts of the calculation may require different choices of half-planes...

(iv) Prove that

$$I_4 = \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2} \ .$$

Hint: remember the freedom granted by Cauchy's theorem...

(v) An integral representation for the Heaviside step function $\theta(x)$ is the following:

$$\theta(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ixz}}{z - i\epsilon} \mathrm{d}z \ ;$$

verify, using the residue theorem, that it gives indeed the usual $\theta(x)$. What happens for $x = 0$? [*Hint*: how is $\log(z)$ defined in $\mathbb{C}$?]