Exercise sheet VI

June 1 [correction: June 8]

Problem 1 [*Free fermion propagator*]

(i) In the Lecture you have been given the following form for the free fermionic propagator (Euclidean formalism):

$$S_F(x,y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p e^{-ip(x-y)} \frac{+i\gamma_\mu p_\mu + m}{p^2 + m^2} \quad .$$

Derive the above expression as the inverse $S_F(x,y) = A^{-1}(x,y)$ of the kinetic operator A appearing in the free action,

$$S[\psi, \overline{\psi}] = \int \mathrm{d}^4 x \int \mathrm{d}^4 y \; \overline{\psi}(x) \underbrace{\delta(x-y)\Big(\gamma_\mu \partial_\mu^{(y)} + m\Big)}_{=A(x,y)} \psi(y)$$

(ii) Carry out the integration over p_0 in the above expression for S_F in order to get the time-momentum representation $S_F(\mathbf{p}, t)$, with $t = x_0 - y_0$.

(iii) Using the above propagator, compute the vacuum expectation value

$$\langle \Omega | \Big(\int \mathrm{d}^3 x \; \psi(\mathbf{x}, t) \Big) \Big(\int \mathrm{d}^3 y \; \psi^\dagger(\mathbf{y}, 0) \Big) | \Omega \rangle$$

and provide a physical interpretation for the result.

Problem 2 [*Faddeev-Popov method*]

Consider the integral

$$I = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \;\; e^{-(x^2+y^2)} \quad .$$

Solve it in a Faddev-Popov-inspired way.

- Exploit the $U(1)$ invariance of the "action" $x^2 + y^2$ through the identity

$$1 = \Delta_{\mathrm{FP}} \int \mathrm{d}g \; \delta\Big(F(x^g, y^g)\Big) \quad ,$$

where g is a 2×2 rotation matrix in the x-y plane,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^g \\ y^g \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix} \quad ,$$

and F must be such that, for any given (x, y), there exists at least one g for which $F(x^g, y^g) = 0$. Parameterize $\int \mathrm{d}g$ appropriately and simplify $\Delta_{\mathrm{FP}}^{-1}$ as much as possible. *[Exercise continues on next page!]*

- Choose now $F(x, y) = y$ and evaluate explicitly Δ_{FP} for this case.
- Use the above to solve

$$I = \int \mathrm{d}x\mathrm{d}y \underbrace{\Delta_{\text{FP}} \int \mathrm{d}g \; \delta\Big(F(x^g, y^g)\Big)}_{=1} e^{-(x^2+y^2)} \quad ,$$

following the steps seen in the Lecture in order to fix the gauge, and verify the final result $I = \pi$.