Exercise sheet VII

June 8 [correction: June 15]

Problem 1 [*Gauge principle, Abelian field*]

The Lagrangian for a free complex scalar field (Euclidean version) reads:

$$\mathscr{L}_0 = \Big(\partial_\mu \phi^*(x)\Big)\Big(\partial_\mu \phi(x)\Big) + m^2 \phi^*(x)\phi(x) \ .$$

- Consider the *global* $U(1)$ symmetry, realised by

$$\begin{cases} \phi & \rightarrow \quad \phi' = U\phi \\ \phi^* & \rightarrow \quad \phi^{*\prime} = \phi^* U^\dagger \end{cases} ,$$

  with $U = e^{i\theta}$ and θ a constant $\in [0, 2\pi)$. Determine the corresponding current J_μ and conserved charge Q.

- Now promote U to a *local* symmetry transformation $U(x) = e^{i\theta(x)}$, i.e. a *gauge transformation*. What happens to the invariance of $\mathscr{L}_0$?

- Show that the variation upon an infinitesimal gauge transformation is of the form

$$\delta\mathscr{L}_0 = (\mathscr{L}_0' - \mathscr{L}_0) = (\partial_\mu \theta) \ J_\mu \ ,$$

  Introduce now a four-vector A_μ which couples to the current according to

$$\mathscr{L}_1 = q J_\mu A_\mu$$

  and identify the transformation law that has to be required for A_μ in order to cancel the $\delta\mathscr{L}_0$ above.

- In the new $\mathscr{L}_0 + \mathscr{L}_1$, however, a new non-gauge-invariant term appears: verify then that with an additional

$$\mathscr{L}_2 \propto A_\mu A_\mu \phi^* \phi$$

  one can achieve $\delta\mathscr{L}_2 = -\delta(\mathscr{L}_0 + \mathscr{L}_1)$, i.e. build a gauge-invariant Lagrangian.

- Finally, A_μ should be a dynamical field: provide its own (gauge-invariant) kinetic term $\mathscr{L}_3[A_\mu]$ (the simplest term containing derivatives of A_μ) and write down the complete Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3$.

⋆ Realise that you have just constructed the theory of electromagnetism !

♢ *Exercises continue on next page !* ♢

♣ *Optional*: Consider now two identical complex scalar fields $\Phi = (\phi_1, \phi_2)$ and a $SU(2)$ symmetry instead of the previous $U(1)$. Proceeding in the same way as before, you will construct "two-colour QCD" (a.k.a. $\mathrm{QC_2D}$).

Problem 2 [*Feynman rules in axial gauge*]

Write down the effective action (gauge-fixing term, ghost term) for QCD (i.e. the $SU(3)$ gauge theory) in the *axial gauge*:

$$n_\mu A^a_\mu = 0 \ , \quad \text{for some fixed } n_\mu \ : \ n_\mu n_\mu = 1 \ ;$$

What happens to the ghost? Derive the gluon propagator (in momentum space) and sketch the corresponding Feynman rules for the quark-gluon interactions.

Hint: when inverting the gluon operator, the symmetry-motivated Ansatz

$$\Big(- k^2 g_{\mu\nu} + k_\mu k_\nu - \frac{1}{\alpha} n_\mu n_\nu \Big)^{-1} = A g_{\mu\nu} + B k_\mu k_\nu + C(k_\nu n_\mu + k_\mu n_\nu)$$

will suffice.