Sheet VIII - ADDENDUM

$\boxed{\text{EX. 2 (i)}}$

- Correct form of spin-rotation operator on spinors:

$$\hat{R}_J(\alpha): \quad \hat{\psi} \longmapsto \exp\left[\alpha\, \varepsilon_{Jkl} \frac{\gamma^k \gamma^l}{4}\right] \hat{\psi}$$

Remember $\quad [\gamma_a, \gamma_b] = -\frac{i}{2} \sigma_{ab}$

Calling $\quad \sigma^{xy} \equiv \sigma^z, \; \sigma^{yz} \equiv \sigma^x, \; \sigma^{zx} \equiv \sigma^y$:

$$\hat{R}_J(\alpha) = \exp\left[-\frac{i\alpha}{8} \sigma^J\right], \quad J = x, y, z$$

- Abstract operation of rotating, $\mathbb{R}_J(\alpha)$, has different REPRESENTATIONS. On the "single spinors space", its representation is the above $\hat{R}_J(\alpha)$, while on a $(\bar{\psi} \cdots \psi)$ state such as $|\phi\rangle$ we cannot write it down explicitly. But, indirectly:

$$\begin{tikzcd}
|\phi\rangle \arrow[rr, dashed, "\mathbb{R}", "?"'] \arrow[d, Rightarrow] & & |\phi\rangle_{\text{rotated}} \\
(\bar{u}\, \gamma_5\, d) \arrow[rr, bend left, "\hat{R}_J"] \arrow[rr, bend right, "\hat{R}_J"'] & & (\bar{u}_{\text{rotated}}\, \gamma_5\, d_{\text{rotated}}) \arrow[u, Rightarrow, "\text{Yes! Now } |\phi\rangle_{\text{rotated}} \text{ can be found}"']
\end{tikzcd}$$

In practice:

- infinitesimal α
- consider now $\mathbb{R}_z$: will give J_z (provided $|\phi\rangle$ is an eigenstate)

$$|\phi\rangle = \hat{\bar{U}}(x)\gamma_5 \hat{d}(x)|\Omega\rangle = u^+ \gamma_0 \gamma_5 d |\Omega\rangle$$

Using $\hat{R}_z(\alpha)$:

$$d \longmapsto \left[1 + \frac{\alpha}{4}(\gamma_x\gamma_y - \gamma_y\gamma_x)\right] d$$

$$u^+ \mapsto u^+\left[1 + \frac{\alpha}{4}\underbrace{(\gamma_x\gamma_y - \gamma_y\gamma_x)^+}_{\underbrace{\gamma_y^+\gamma_x^+ - \gamma_x^+\gamma_y^+}_{\gamma_y\gamma_x - \gamma_x\gamma_y}}\right]$$

$$\begin{cases} \gamma_i^+ = -\gamma_i \\ \gamma_0^+ = +\gamma_0 \end{cases} \quad !$$

Then:

$$|\phi\rangle \overset{R_z}{\longmapsto} u^+\left[1 + \frac{\alpha}{4}(\gamma_y\gamma_x - \gamma_x\gamma_y)\right]\gamma_0\gamma_5\left[1 + \frac{\alpha}{4}(\gamma_x\gamma_y - \gamma_y\gamma_x)\right] d|\Omega\rangle =$$

$$\gamma_x\gamma_y\gamma_0\gamma_5 = \gamma_0\gamma_x\gamma_y\gamma_5 = \gamma_0\gamma_5(\gamma_x\gamma_y) \quad \text{and so on}$$

$$= u^+\gamma_0\gamma_5\left[1 + \frac{\alpha}{4}(\gamma_y\gamma_x - \gamma_x\gamma_y + \gamma_x\gamma_y - \gamma_y\gamma_x) + \mathcal{O}(\alpha^2)\right] d|\Omega\rangle =$$

$$= |\phi\rangle + \mathcal{O}(\alpha^2)$$

$$\mathbb{R}_z : |\phi\rangle \longmapsto |\phi'\rangle = |\phi\rangle + \mathcal{O}(\alpha^2)$$

since $\mathbb{R}_z : |\phi\rangle \mapsto e^{i\alpha \hat{J}_z}|\phi\rangle$, we are infinitesimal and the final result is:

$$\boxed{|\phi\rangle \text{ has } J_z = 0}$$

Exactly the same procedure leads to:

$$J_x = J_y = 0$$

- Immediately, $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$:

$$\boxed{|\phi\rangle \text{ has } J^2 = 0}$$

It is a particular case, in that $\hat{J}_{x,y,z}$ cannot in general be simultaneously diagonal, <u>except</u> when they are all <u>zero</u>.

Ex. 2 (ii)

(*) Side note: in isospin space, the doublet $\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$ transforms as the $\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$. Indeed: $(\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3))$.

A) $\begin{pmatrix} u \\ d \end{pmatrix} \mapsto e^{i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u' \\ d' \end{pmatrix}$

B) $\begin{pmatrix} u \\ d \end{pmatrix}^{+} \mapsto \begin{pmatrix} u \\ d \end{pmatrix}^{+} e^{-i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}} = \begin{pmatrix} u' \\ d' \end{pmatrix}^{+}$

insert $\sigma_2^2 = \mathbb{1}$, right-multiply by σ_2:

$$\begin{pmatrix} u' \\ d' \end{pmatrix}^{+} \sigma_2 = \begin{pmatrix} u \\ d \end{pmatrix}^{+} \sigma_2 \sigma_2 \, e^{-i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}} \, \sigma_2 \qquad \boxed{\cdot}$$

the σ_2, passing through the exp, flips sign to σ_1, σ_3 only:

remember that

$$e^{-i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}} = \mathbb{1}\left(\cos\frac{|\bar{\alpha}|}{2}\right) + \left(\frac{i}{|\bar{\alpha}|}\sin\frac{|\bar{\alpha}|}{2}\right)(\alpha_i\sigma_i)$$

then

$$\sigma_2\, e^{-i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}}\,\sigma_2 = \sigma_2\mathbb{1}\sigma_2\left(\cos\frac{|\bar{\alpha}|}{2}\right) + \left(\frac{i}{|\bar{\alpha}|}\sin\frac{|\bar{\alpha}|}{2}\right)\left[\alpha_i\underbrace{(\sigma_2\sigma_i\sigma_2)}\right]$$

since $\{\sigma_a,\sigma_b\} = 2\delta_{ab}\mathbb{1}$,

$$\sigma_2\sigma_1\sigma_2 = -\sigma_2\sigma_2\sigma_1 = -\sigma_1$$

$$\sigma_2\sigma_3\sigma_2 = -\sigma_2\sigma_2\sigma_3 = -\sigma_3$$

but $\sigma_2\sigma_2\sigma_2 = +\sigma_2$

One then writes:

$$\sigma_2\, e^{-i\bar{\alpha}\cdot\frac{\bar{\sigma}}{2}}\,\sigma_2 = e^{-i\frac{\bar{\alpha}}{2}\cdot(-\sigma_1,+\sigma_2,-\sigma_3)}$$

so $\boxed{\cdot}$ becomes

$$\begin{pmatrix} u' \\ d' \end{pmatrix}^{+}\sigma_2 = \begin{pmatrix} u \\ d \end{pmatrix}^{+}\sigma_2\, e^{-i\frac{\bar{\alpha}}{2}\cdot(-\sigma_1,+\sigma_2,-\sigma_3)}$$

$$i\begin{pmatrix} d'^{+} \\ -u'^{+} \end{pmatrix}^{T} = i\begin{pmatrix} d^{+} \\ -u^{+} \end{pmatrix}^{T} e^{-i\frac{\bar{\alpha}}{2}\cdot(-\sigma_1,+\sigma_2,-\sigma_3)}$$

get rid of the i, <u>transpose</u> ($\sigma_1^T = +\sigma_1$, $\sigma_2^T = -\sigma_2$, $\sigma_3^T = +\sigma_3$!):

$$\begin{pmatrix} d'^{+} \\ -u'^{+} \end{pmatrix} = e^{i\frac{\alpha_i}{2}\sigma_i}\begin{pmatrix} d^{+} \\ -u^{+} \end{pmatrix}$$

with the γ_0 on the right:

$$\begin{pmatrix} \bar{d}' \\ -\bar{u}' \end{pmatrix} = e^{i\frac{\bar{\alpha}}{2}\cdot\bar{\sigma}}\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} \Longrightarrow \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} \text{ TRANSFORMS AS } \begin{pmatrix} u \\ d \end{pmatrix}\ !$$

Using the "spin notation" (for isospin space)

$$\bar{d} \longleftrightarrow \uparrow$$

$$-\bar{u} \longleftrightarrow \downarrow$$

one sees that the state

$$|\phi\rangle \leftrightsquigarrow \bar{u} d \leftrightsquigarrow \downarrow\downarrow$$

can only mean TRIPLET

$$I_3 = -1$$

$$I = 1$$

(it is not so, for example, with $\bar{u}u$ ...)

Actual I_3 calculation:

$$\hat{\tilde{R}}_J(\alpha) = \exp(i\alpha \hat{I}_J) \quad \text{on quark doublets:}$$

$$\hat{\tilde{R}}_J(\alpha)\left[\begin{pmatrix}\hat{u}\\ \hat{d}\end{pmatrix}\right] = \exp\left(i\alpha \frac{\sigma_J}{2}\right)\begin{pmatrix}\hat{u}\\ \hat{d}\end{pmatrix}$$

Again, the representation of $\tilde{R}_J$ on $|\phi\rangle$ is not directly accessible: one proceeds similarly as for the spin.

Choose $J \equiv z$.

Drop the (trivial) x-dependence, and write the state as

$$|\phi\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\dagger} \gamma_0 \gamma_5 \begin{pmatrix}0\\1\end{pmatrix} |\Omega\rangle \,, \quad \text{using the basis "}\begin{pmatrix}u\\d\end{pmatrix}\text{"}: \begin{cases}\hat{u} = \begin{pmatrix}1\\0\end{pmatrix}\\ \hat{d} = \begin{pmatrix}0\\1\end{pmatrix}\end{cases}$$

and act on $|\phi\rangle$ with $\hat{\tilde{R}}_3(\alpha) = \exp\left(i\frac{\alpha}{2}\sigma_3\right)$

⑥

the rotation gives:

$$\begin{pmatrix}1\\0\end{pmatrix}^{\dagger} \gamma_0 \gamma_5 \begin{pmatrix}0\\1\end{pmatrix}|\Omega\rangle \xmapsto{\hat{\tilde{R}}_3} \left[e^{i\frac{\alpha}{2}\sigma_3}\begin{pmatrix}1\\0\end{pmatrix}\right]^{\dagger}\gamma_0\gamma_5\left[e^{i\frac{\alpha}{2}\sigma_3}\begin{pmatrix}0\\1\end{pmatrix}\right]|\Omega\rangle =$$

(infinitesimal...) $$\simeq \left[\left(1+i\frac{\alpha}{2}\begin{pmatrix}1&\\&-1\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix}\right]^{\dagger}\gamma_0\gamma_5\left[\left(1+\frac{i\alpha}{2}\begin{pmatrix}1&\\&-1\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}\right]|\Omega\rangle =$$

$$= \left[\left(1+\frac{i\alpha}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}\right]^{\dagger}\gamma_0\gamma_5\left[\left(1-\frac{i\alpha}{2}\right)\begin{pmatrix}0\\1\end{pmatrix}\right]|\Omega\rangle =$$

$$= \left(1-2\frac{i\alpha}{2}\right)\left[\begin{pmatrix}1\\0\end{pmatrix}^{\dagger}\gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix}\right]|\Omega\rangle = \underbrace{(1-i\alpha)\,|\phi\rangle}_{\text{this has to be}}$$

$(1+i\alpha I_3)|\phi\rangle$

if eigenstate (it is)

then

$$\boxed{|\phi\rangle \text{ has } I_3 = -1}$$

Now, a combination of quark and antiquark can be only singlet $(I=0)$ or triplet $(I=1)$, but since $I_3=-1$

$$\rightarrow \boxed{I = 1}$$

(i.e. $2 \otimes 2 = 1 \oplus 3$ and nothing else...

$1 \downarrow I=0$, $3 \downarrow I=1$)

Proper way to get I

Ingenious trick: consider two rotations on the same direction (still infinitesimal)

$$e^{i\alpha \hat{I}_x} e^{i\beta \hat{I}_x} = (\mathbb{1} + i\alpha \hat{I}_x + \dots)(\mathbb{1} + i\beta \hat{I}_x + \dots) =$$

$$= (\mathbb{1} + i(\alpha+\beta)\hat{I}_x - \alpha\beta \hat{I}_x^2 + \dots)$$

Then, by comparing to the $\mathcal{O}(\alpha\beta)$ term of

$$\hat{\tilde{R}}_{I_x}(\alpha)\left[\hat{\tilde{R}}_{I_x}(\beta)\left(|\phi\rangle\right)\right]$$

one gets $\hat{I}_x^2$, that is, how it acts on $|\phi\rangle$.

- Repeat for $\hat{I}_y^2$ and (in principle, but here we have an eigenstate so it is already known) $\hat{I}_z^2$ $(= (-1)^2$, see p. 6$)$: at the end one finds the action of

$$|\phi\rangle \xmapsto{\hat{I}_x^2 + \hat{I}_y^2 + \hat{I}_z^2} |\text{something}\rangle$$

from which I^3 (the eigenvalue) is read.

In practice:

- first rotation, $|\phi\rangle \longmapsto |\phi^1\rangle = \hat{\tilde{R}}_x(\beta)\left(|\phi\rangle\right)$

$$|\phi^1\rangle = \Big[\big(\underbrace{1+i\tfrac{\beta}{2}\sigma_x}_{\boxed{=\begin{pmatrix}0&1\\1&0\end{pmatrix}}}\big)\begin{pmatrix}1\\0\end{pmatrix}\Big]^\dagger \gamma_0\gamma_5 \Big[\big(1+i\tfrac{\beta}{2}\sigma_x\big)\begin{pmatrix}0\\1\end{pmatrix}\Big]|\Omega\rangle =$$

$$= \Big[\begin{pmatrix}1\\0\end{pmatrix} + \frac{i\beta}{2}\begin{pmatrix}0\\1\end{pmatrix}\Big]^\dagger \gamma_0\gamma_5 \Big[\begin{pmatrix}0\\1\end{pmatrix} + \frac{i\beta}{2}\begin{pmatrix}1\\0\end{pmatrix}\Big]|\Omega\rangle =$$

$$= |\phi\rangle + \frac{i\beta}{2}\Big\{\begin{pmatrix}1\\0\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix}\Big\}|\Omega\rangle + \mathcal{O}(\beta^2)$$

*** this shows $|\phi\rangle$ is NOT eigenstate for $\hat{\tilde{R}}_x$, as was expected ***

- second rotation, $|\phi^1\rangle \longmapsto |\phi^2\rangle = \hat{\tilde{R}}_x(\alpha)\left(|\phi^1\rangle\right)$: using

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \big(1+\tfrac{i\alpha}{2}\sigma_x\big)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \big(1+\tfrac{i\alpha}{2}\sigma_x\big)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

one finds

$$|\phi^1\rangle \mapsto \Big[\begin{pmatrix}1\\0\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}\Big]^\dagger \gamma_0\gamma_5 \Big[\begin{pmatrix}0\\1\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}1\\0\end{pmatrix}\Big]|\Omega\rangle$$

$$+ \frac{i\beta}{2}\Big\{\Big[\begin{pmatrix}1\\0\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}\Big]^\dagger \gamma_0\gamma_5 \Big[\begin{pmatrix}1\\0\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}\Big] - \Big[\begin{pmatrix}0\\1\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}1\\0\end{pmatrix}\Big]^\dagger \gamma_0\gamma_5 \Big[\begin{pmatrix}0\\1\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}1\\0\end{pmatrix}\Big]\Big\}|\Omega\rangle$$

The $\sim\alpha\beta$ term, from above, is

$$\frac{i\beta}{2}\Big\{\begin{pmatrix}1\\0\end{pmatrix}^\dagger \gamma_0\gamma_5 \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix} - \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}1\\0\end{pmatrix} + \frac{i\alpha}{2}\begin{pmatrix}1\\0\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix} - \frac{i\alpha}{2}\begin{pmatrix}0\\1\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}1\\0\end{pmatrix}\Big\}|\Omega\rangle =$$

$$= -\frac{\alpha\beta}{4}\Big\{2\begin{pmatrix}1\\0\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}^\dagger \gamma_0\gamma_5\begin{pmatrix}1\\0\end{pmatrix}\Big\}|\Omega\rangle =$$

$$= -\alpha\beta\Big[\frac{1}{2}\hat{u}\gamma_5\hat{d} - \frac{1}{2}\hat{d}\gamma_5\hat{u}\Big]|\Omega\rangle \quad \blacktriangle$$

*** Again, something more than $\propto |\phi\rangle$ is found (not an eigenstate) ***

Ⓐ must be compared to (p.7): then,

$$\hat{I}_x^2\,|\phi\rangle = \frac{1}{2}|\phi\rangle - \frac{1}{2}\left(\hat{\bar{d}}\,\gamma_5\,\hat{u}\right)|\Omega\rangle$$

Now for $\hat{I}_y^2$: the procedure is exactly the same, with

$$\hat{\tilde{R}}_y(\beta) : \begin{cases} \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix} - \frac{\beta}{2}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix} + \frac{\beta}{2}\begin{pmatrix}1\\0\end{pmatrix} \end{cases}$$

one finds

$$|\phi\rangle \xmapsto{\hat{\tilde{R}}_y(\beta)} |\phi\rangle + \frac{\beta}{2}\underbrace{\left\{ \begin{pmatrix}1\\0\end{pmatrix}^{\dagger}\gamma_0\gamma_5\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}^{\dagger}\gamma_0\gamma_5\begin{pmatrix}0\\1\end{pmatrix} \right\}|\Omega\rangle}$$

$$\Big\downarrow\ \hat{\tilde{R}}_y(\alpha)$$

$$\cdots\cdots - \alpha\beta\left\{ \frac{1}{2}|\phi\rangle + \frac{1}{2}\,\hat{\bar{d}}\,\gamma_5\,\hat{u}\,|\Omega\rangle \right\} + \cdots\cdots$$

hence $$\hat{I}_y^2\,|\phi\rangle = \frac{1}{2}|\phi\rangle + \frac{1}{2}\,\hat{\bar{d}}\,\gamma_5\,\hat{u}\,|\Omega\rangle$$

Last, we know already that

$$\hat{I}_z^2\,|\phi\rangle = (-1)^2\,|\phi\rangle = |\phi\rangle$$

so, finally,

$$\hat{I}^2\,|\phi\rangle = \left(1 + \frac{1}{2} + \frac{1}{2}\right)|\phi\rangle + \cancel{\left(\frac{1}{2} - \frac{1}{2}\right)\left[\hat{\bar{d}}\,\gamma_5\,\hat{u}\,|\Omega\rangle\right]} = 2\,|\phi\rangle$$

From spin theory, remember

$$\hat{I}^2\,|\phi\rangle = I(I+1)\,|\phi\rangle$$

→ hence $I = 1$ ■

EX. 2 (vi)

For the state $|\phi'\rangle = \hat{\bar{u}}(x)\, \gamma_J\, \hat{d}(x)\, |\Omega\rangle$,

everything proceeds as for $|\phi\rangle$, except the SPIN.

$|\phi'\rangle$ is actually a triplet $|\phi'_J\rangle$, $J = 1,2,3$: one should examine the effect of rotations R_K on $|\phi'_J\rangle$ ("$3\times 3 = 9$ calculations")

- Effect of R_z (cf. page 2), i.e. $K = z$:

$$|\phi'_J\rangle \xmapsto{R_z} u^+\left(1+\frac{\alpha}{4}(\gamma_y\gamma_x - \gamma_x\gamma_y)\right)\gamma_0\gamma_J\left(1+\frac{\alpha}{4}(\gamma_x\gamma_y - \gamma_y\gamma_x)\right) d\, |\Omega\rangle =$$

(neglect $\mathcal{O}(\alpha^2)\ldots$)
$$= u^+\gamma_0\gamma_J\, d\, |\Omega\rangle + \frac{\alpha}{4}\, u^+ \underbrace{\left[(\gamma_y\gamma_x - \gamma_x\gamma_y)\gamma_0\gamma_J + \gamma_0\gamma_J(\gamma_x\gamma_y - \gamma_y\gamma_x)\right]} d\,|\Omega\rangle$$

This $[\ldots] = \gamma_0\left\{(\gamma_y\gamma_x - \gamma_x\gamma_y)\gamma_J + \gamma_J(\gamma_x\gamma_y - \gamma_y\gamma_x)\right\} =$

$\boxed{J=z}$ $= 0$, and:

$$|\phi'_z\rangle \xmapsto{R_z} |\phi'_z\rangle \ : \text{ it has } J_z = 0 \ \blacksquare$$

$\boxed{J=x}$ $= 4\gamma_0\gamma_y$ and:

$$|\phi'_x\rangle \xmapsto{R_z} |\phi'_x\rangle + \alpha\, u^+\gamma_0\gamma_y\, d\, |\Omega\rangle =$$

$$= |\phi'_x\rangle + \alpha\, |\phi'_y\rangle$$

i.e. not eigenstate.

$\boxed{J=y}$ similarly,

$= -4\gamma_0\gamma_x$:

$$|\phi'_y\rangle \xmapsto{R_z} |\phi'_y\rangle - \alpha\, |\phi'_x\rangle$$

$\rightarrow$ In the basis $\{|\phi'_x\rangle, |\phi'_y\rangle, |\phi'_z\rangle\}$, the $R_z(\alpha)$ acts on 3-component vectors $|\vec{\phi}'\rangle$ as:

$$R_z(\alpha)\,|\vec{\phi}'\rangle = \begin{pmatrix} 1 & -\alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} |\vec{\phi}'\rangle \qquad \text{(infinitesimal)}$$

i.e.

$$R_z(\alpha) = \mathbb{1} + i\alpha \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \text{cf. } \mathbb{1} + i\alpha \hat{J}_z$$

$$\boxed{\Longrightarrow \hat{J}_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$|\phi'_z\rangle$ is an eigenstate of R_z, while $|\phi'_{x,y}\rangle$ are <u>NOT</u>.

But one can diagonalise: build

$$|\phi'_{-1}\rangle = \frac{i}{\sqrt{2}}\left(|\phi'_x\rangle + i|\phi'_y\rangle\right)$$

$$|\phi'_{+1}\rangle = \frac{1}{\sqrt{2}}\left(|\phi'_x\rangle - i|\phi'_y\rangle\right)$$

and in the basis $\{|\phi'_{+1}\rangle, |\phi'_z\rangle, |\phi'_{-1}\rangle\}$ it is:

$$\boxed{\hat{J}_z^N = \begin{pmatrix} +1 & & \\ & 0 & \\ & & -1 \end{pmatrix}}$$

Just what one expects from a $J=1$ su(2) representation. (SPIN-ONE TRIPLET)

- Under the cyclic permutation $x \to y \to z$ one gets immediately, in the "old" x/y/z basis,

$$\hat{J}_x = \begin{pmatrix} 0 & & \\ & & i \\ & -i & \end{pmatrix}, \qquad \hat{J}_y = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}$$

So, in the "old" basis, there are

$$\hat{J}_x = \left(\begin{array}{c|cc} 0 & & \\ \hline & & i \\ & -i & \end{array}\right), \quad \hat{J}_y = \left(\begin{array}{c|c|c} & & -i \\ \hline & & \\ \hline i & & \end{array}\right), \quad \hat{J}_z = \left(\begin{array}{cc|c} & i & \\ -i & & \\ \hline & & \end{array}\right)$$

which in the "new" basis become

$$\hat{J}_x^N = \frac{1}{\sqrt{2}}\begin{pmatrix} & -1 & \\ -1 & & i \\ & -i & \end{pmatrix}, \quad \hat{J}_y^N = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -1 \\ & -1 & \end{pmatrix}, \quad \hat{J}_z^N = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

- They are indeed a representation of the $J=1$ triplet: one can verify

$$[\hat{J}_x, \hat{J}_y] = -i\,\hat{J}_z$$

$$[\hat{J}_y, \hat{J}_z] = -i\,\hat{J}_x$$

$$[\hat{J}_z, \hat{J}_x] = -i\,\hat{J}_y$$

- One can build the quadratic Casimir, that - as it must be - turns out to be $\propto$ identity:

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = 2\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \underbrace{J(J+1)}_{\text{hence, } J=1\,!} \cdot \mathbb{1}$$