Exercise sheet XI

July 6 [correction: July 13]

Problem 1 [*Fermions on the lattice*]

Consider, in the following, a $(1+1)$-dimensional Minkowski spacetime. Here the metric tensor is $g_{\mu\nu} = \mathrm{diag}(+1, -1)$, the two 2×2 gamma-matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and spinors have two components.

(i) Solve the Dirac equation on the continuum

$$i\gamma_0 \partial_0 \psi(x,t) + i\gamma_1 \partial_x \psi(x,t) - m\psi(x,t) = 0$$

up to identifying the dispersion relation $E = E(p)$. To do so:

- get rid of derivatives with the plane-wave Ansatz

$$\psi(x,t) = e^{i(Et-px)} u(p,E) \ ;$$

- get rid of the gamma-matrix structure with the Ansatz

$$u(p,E) = (-\gamma_0 E + \gamma_1 p + m)v \ ,$$

  with v some two-component spinor.

How many solutions p do you find for a given energy $E > m$?

(ii) Now discretise *only* the spatial direction x with a spacing a, choosing a symmetric derivative for rewriting the Dirac equation:

$$\partial_x \psi(x) \quad \longmapsto \quad \frac{\psi(x+a) - \psi(x-a)}{2a} \ .$$

Proceed as in the previous case, with an adequate $u \to v$ Ansatz (*Hint*: is it still true that $-\infty < p < +\infty$?), and write down the lattice version of the dispersion $E = E_{\mathrm{lat}}(p)$. Compare it with the continuum result: in particular, how many solutions (for a given energy E) do you find on the lattice?

(iii) Repeat the above exercise, but now with the following discretised derivative:

$$\partial_x \psi(x) \quad \longmapsto \quad \frac{\psi(x+a) - \psi(x)}{a} \ .$$

Which essential property of the continuum action is lost in this case?