Exercise sheet 0 (special session)

April 24, 2013

Problem 1 [*Functional differentiation*] : The formal definition of the functional derivative is as follows: with $F[\phi]$ a functional of $\phi(x)$,

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\mathrm{d}}{\mathrm{d}s} F[\phi(x) + s\delta(x-y)]\Big|_{s=0} \quad .$$

(i) In calculating the transition probability $\langle x_2, t_2 | x_1, t_1\rangle$ for the harmonic oscillator,

$$S[x] = \int_{t_1}^{t_2} \mathrm{d}u \left(\frac{m}{2}\dot{x}^2(u) - \frac{m\omega^2}{2} x^2(u)\right) \quad ; \quad x = x(t) \quad ,$$

one can write the trajectory as $x(t) = x_{\mathrm{cl}}(t) + y(t)$, with boundary conditions $x_{\mathrm{cl}}(t_1) = x_1$, $x_{\mathrm{cl}}(t_2) = x_2$ and $y(t_1) = y(t_2) = 0$.

Expanding around the classical solution to the equations of motion, x_{cl}, the following exact result holds:

$$S[x] = S[x_{\mathrm{cl}}] + \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{m}{2}\dot{y}^2(t) - \frac{m\omega^2}{2} y^2(t)\right) \quad .$$

(a) Show, using the formal definition above, that indeed

$$\frac{\delta S[x]}{\delta x(t)} = -m\ddot{x}(t) - m\omega^2 x(t) \quad .$$

(b) Show that

$$\frac{\delta^2 S[x]}{\delta x(t)\delta x(t')} = \delta(t-t')\Big(-m\frac{\mathrm{d}^2}{\mathrm{d}t'^2} - m\omega^2\Big) \quad .$$

(c) Show that, for any $n > 2$,

$$\frac{\delta^n S[x]}{\delta x(t)\delta x(t')\cdots\delta x(t^{(n-1)})} = 0 \quad ;$$

for which class of potentials do you expect this result to hold?

(ii) Starting from the real Klein-Gordon action in four dimensions, with metric $g^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$,

$$S[\phi] = \int \mathrm{d}^4x \Big(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi^2(x)\Big) \quad ,$$

evaluate the derivative

$$\frac{\delta S[\phi]}{\delta\phi(x)}$$

and verify that setting it to zero yields the Klein-Gordon equation $(\Box - m^2)\phi(x) = 0$. [*Hint*: in integration by parts, the finite contribution can be neglected.]

Problem 2 [*Contour integration, residues*] :

(i) Prove that

$$I_1 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 3\cos\theta} = \frac{\pi}{2} \quad .$$

(ii) Prove that

$$I_2 = \int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \mathrm{d}x = \frac{\pi}{6} \quad .$$

(iii) Prove that

$$I_3 = \int_{-\infty}^{+\infty} \frac{\cos x}{1 + x^2} \mathrm{d}x = \frac{\pi}{e} \quad .$$

Hint: different parts of the calculation may require different choices of half-planes...

(iv) Prove that

$$I_4 = \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2} \quad .$$

Hint: remember the freedom granted by Cauchy's theorem...

(v) An integral representation for the Heaviside step function $\theta(x)$ is the following:

$$\theta(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ixz}}{z - i\epsilon} \mathrm{d}z \quad ;$$

verify, using the residue theorem, that it gives indeed the usual $\theta(x)$. What happens for $x = 0$? [*Hint*: how is $\log(z)$ defined in $\mathbb{C}$?]