Exercise sheet I

April 24 [correction: May 8]

Problem 1 [*Zassenhaus formula*] :

Using the Zassenhaus formula $e^{A+B} = e^A e^B e^{Z_2} e^{Z_3} e^{Z_4} \ldots$ with $Z_2 = \frac{1}{2}[B,A]$, $Z_3 = -\frac{1}{6}\left(2[B,[B,A]] + [A,[B,A]]\right)$ and exponents $Z_n, n > 3$ containing higher powers of A and B, show the relation

$$e^{-i\epsilon H} = e^{-i\epsilon(H_0+V)} = e^{-i\epsilon V/2} e^{-i\epsilon H_0} e^{-i\epsilon V/2} + \mathcal{O}(\epsilon^3)\,. \tag{1}$$

Problem 2 [*Harmonic oscillator – partition function*] :

You are to obtain the partition function of the (quantum mechanical and classical) harmonic oscillator in three different ways:

(i) Starting from the formula of the harmonic oscillator seen in the Lectures

$$\langle x,t|x,0\rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega t)}} \exp\left(i\frac{x^2 m\omega}{\sin(\omega t)}(\cos(\omega t) - 1)\right) \tag{2}$$

go to Euclidean time ($t = -i\tau$) and calculate the partition function $Z = \int \mathrm{d}x\ \langle x|e^{-\tau H}|x\rangle$.

(ii) The oscillator spectrum is discrete. Make use of this fact and calculate $Z = \sum_n e^{-\tau E_n}$. Compare to (i).

(iii) Now calculate the partition function of the classical oscillator using $Z = \int \mathrm{d}x\mathrm{d}p\, \mathrm{e}^{-\tau(T+V)}$.

Compute the free energy $F = -1/\tau \log Z$ for the three cases and show that they agree for $\tau \ll 1$. Why?

Problem 3 [*Functional derivative, chain rule*] :

Calculate the functional derivative, with respect to $f(x)$, of the functional $F[g(f)] = \int \mathrm{d}x\, g(f(x))$ with $g(f) = (\frac{\mathrm{d}}{\mathrm{d}x} f(x))^n$ using:

(i) the definition of the (functional) chain rule $\frac{\delta F[g(\phi)]}{\delta\phi(y)} = \int \mathrm{d}s \frac{\delta F[g(\phi)]}{\delta g(\phi(s))} \frac{\delta g(\phi(s))}{\delta \phi(y)}$;

(ii) the chain rule of ordinary differential calculus and the equivalent definition $\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\mathrm{d}}{\mathrm{d}s} F[\phi(x) + s\delta(x-y)]\Big|_{s=0}$.