Exercise sheet III, IV

April 29 [solution: May 5-12]

Problem 1 [*Spectral decomposition of an operator*] One sometimes needs to define a function f of an operator (matrix). If the function $f : \mathbb{R} \to \mathbb{R}$ has a Taylor expansion:

$$f(x) = f(0) + f'(0) + \frac{1}{2!} f''(0) + \ldots, \tag{1}$$

then one can use this expansion as the definition of the matrix-valued function $f(A)$ by just replacing $x \to A$. For example,

$$e^A = \mathbb{1} + A + \frac{1}{2!} A^2 + \ldots. \tag{2}$$

However, a more clever way exists, the so-called spectral decomposition of a matrix:

$$f(A) = \sum_\lambda f(\lambda) |\lambda\rangle\langle\lambda|, \tag{3}$$

where λ are eigenvalues of the matrix A and $|\lambda\rangle$ their corresponding eigenvectors.

Consider particular examples:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4}$$

Both A and B have eigenvalues 0, 2. Calculate e^A and e^B by using both prescriptions – Eqs. (2) and (3) – and show that they both lead to the same result.

Problem 2 [*Generalizations of the Gaussian integral*] Calculate the following integrals and check that your answer matches the one given below:

$$G_1 \equiv \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi}, \tag{5}$$

$$G_2 \equiv \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}, \tag{6}$$

$$\langle x^{2n} \rangle \equiv \frac{G_3}{G_2} \equiv \frac{1}{G_2} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} x^{2n} = \frac{(2n-1)!!}{a^n}, \tag{7}$$

$$\frac{G_4}{G_2} \equiv \frac{1}{G_2} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + Jx} = e^{\frac{J^2}{2a}}, \tag{8}$$

compute $\langle x^{2n} \rangle$ again, by using the result for G_4:

$$\langle x^{2n} \rangle = \left(\frac{d}{dJ}\right)^{2n} \frac{G_4}{G_2}\Big|_{J=0} = \left(\frac{d}{dJ}\right)^{2n} e^{\frac{J^2}{2a}}\Big|_{J=0} \tag{9}$$

(J is set to zero at the very end),

$$\frac{G_5}{G_2} \equiv \frac{1}{G_2} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+iJx} = e^{-\frac{J^2}{2a}}, \tag{10}$$

$$\frac{G_6}{G_2} \equiv \frac{1}{G_2} \int_{-\infty}^{\infty} dx e^{\frac{1}{2}iax^2+iJx} = i^{1/2} e^{-i\frac{J^2}{2a}}. \tag{11}$$

Now we want to generalize to N variables $x_1, \ldots, x_N$. Thus, instead of a, we will take a real symmetric $N \times N$ matrix A_{ij} and x, J will be vectors x_i, J_i:

$$G_7 \equiv \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N e^{-\frac{1}{2}xAx} = \sqrt{\frac{(2\pi)^N}{\det A}}, \tag{12}$$

where $xAx \equiv x_i A_{ij} x_j$. To do this integral, you should diagonalize A by an orthogonal transformation that will factorize the integral into N Gaussian integrals like G_4. Generalizations including a source term:

$$G_8 \equiv \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N e^{-\frac{1}{2}xAx+Jx} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}JA^{-1}J}, \tag{13}$$

where $Jx \equiv J_i x_i$,

$$G_9 \equiv \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N e^{\frac{i}{2}xAx+iJx} = \sqrt{\frac{(2\pi i)^N}{\det A}} e^{-\frac{i}{2}JA^{-1}J}. \tag{14}$$

Finally, the analogue of $\langle x^{2n} \rangle$ for many variables:

$$\langle x_i x_j \ldots x_k x_l \rangle \equiv \frac{G_{10}}{G_7} \equiv \frac{1}{G_7} \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N e^{-\frac{1}{2}xAx} x_i x_j \ldots x_k x_l = \sum_{\text{Wick}} (A^{-1})_{ab} \ldots (A^{-1})_{cd}, \tag{15}$$

where the set of indices $\{a, b, \ldots, c, d\}$ represents a permutation of the set of indices $\{i, j, \ldots, k, l\}$, which can be given the name *Wick contractions*. The sum is over all such permutations (contractions).

Work out the following cases:

(i) $\langle x_i x_j \rangle = (A^{-1})_{ij}$,

(ii) $\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{il}(A^{-1})_{jk} + (A^{-1})_{ik}(A^{-1})_{jl}$,

(iii) $\langle x_i x_j x_k x_l x_m x_n \rangle$.

Convince yourself that the above cases reduce to G_3 for $N = 1$ (i.e. a single variable x) and $n = 1, 2, 3$ (i.e. $\langle x^2 \rangle$, $\langle x^4 \rangle$, $\langle x^6 \rangle$).

Problem 3 [*Free scalar field theory*] Consider the free real scalar field theory with the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{16}$$

(i) Write the generating functional $Z[J]$.

(ii) Compute it explicitly, i.e. solve the appropriate Gaussian integral.

Let us consider an arbitrary n-point Green's function of the theory:

$$G_n(x_1, \dots, x_n) \equiv \langle \Omega | T\{\phi(x_n) \dots \phi(x_1)\} |\Omega\rangle = \frac{1}{Z}\int D\phi\, \phi(x_n) \dots \phi(x_1) e^{iS[\phi]}, \tag{17}$$

where we have defined $Z \equiv \int D\phi\, e^{iS[\phi]}$. It is related to the generating functional via:

$$G_n(x_1, \dots, x_n) = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)}\Big|_{J=0}. \tag{18}$$

(i) Calculate the propagator, i.e. $i\langle \Omega | T\{\phi(x)\phi(y)\} |\Omega\rangle$ (we are in free theory, so $|\Omega\rangle = |0\rangle$).

(ii) Calculate the 4-point function $G_4(x_1, x_2, x_3, x_4)$ and draw the corresponding Feynman diagrams.

(iii) Convince yourself (or calculate) that the 6-point function $G_6(x_1, \dots, x_6)$ comes out as expected and draw the corresponding Feynman diagrams.

(iv) What happens if one calculates an n-point function with n odd?

Problem 4 [*Attractive Yukawa potential from scalar exchange*] In this exercise, we will derive the potential between two static point charges (external sources), coupled to a scalar field with Lagrangian density (16). We take $J(x) = J_1(x) + J_2(x)$, with $J_i(x) = \delta^{(3)}(\vec{x} - \vec{x_i})$, and write the generating functional $W[J]$ (generator of connected diagrams, defined as $Z[J] \equiv Z e^{iW[J]}$, where Z was defined below Eq. (17)) for this theory:

$$W[J] = -\frac{1}{2}\int d^4x d^4y J(x) D(x-y) J(y). \tag{19}$$

(i) Insert the expression for the Feynman propagator into Eq. (19), together with the expression for J. Since $J(x) = J_1(x) + J_2(x)$, you will have four terms. Ignore the ones with $J_1 J_1$ and $J_2 J_2$ (they correspond to emission and re-absorption of a particle by the same source, so they don't contribute to the force we want to compute).

(ii) Now consider the term W_{12}, containing $J_1 J_2$ (the term W_{21} is equal, so multiply the term W_{12} by 2) and plug in the Dirac delta functions. Perform the integrations over d^3x and d^3y.

(iii) Perform the integral over dy_0 and use the resulting Dirac delta $\delta(k_0)$ to do the integral over dk_0. The denominator will now always be positive, so $i\epsilon$ from the Feynman propagator can be dropped. Perform also the dx_0 integral, which gives you a factor of T – the temporal extent of the system.

(iv) Now, use the fact that $Z[J] = \langle 0|e^{-iHT}|0\rangle = e^{-iET}$. We then see that $W[J] = -ET$ and hence we have the expression for the potential energy between the two static sources. Define $\vec{r} \equiv \vec{x_2} - \vec{x_1}$ and perform the final integral (over d^3k), going to spherical coordinates (hint: pick up the pole in the upper half-plane).

In the end, you should obtain:

$$E = -\frac{1}{4\pi r}e^{-mr}, \tag{20}$$

i.e. the two sources attract each other by their coupling to the scalar field. Moreover, the attraction decreases exponentially with the distance – the effective range of the interaction is of order $1/m$. Such mechanism was the one that Yukawa proposed in 1935 to describe the attraction between nucleons in atomic nuclei. Knowing the range of the nuclear force, he could then predict the mass of the force carrier, named a meson, to be around 100 MeV. Today we identify this particle with the pion, with a mass of approx. 135 MeV (neutral pion) or 140 MeV (charged pion).

Bonus problem: what happens in a general d-dimensional space-time?