Exercise Sheet VI

May 27 [Solution June 02]

Problem 1. *The Fermion Propagator*

In this exercise we will closer examine the free fermionic theory to get used to working with fermions as opposed to bosons.

- Calculate the Gaussian integral

$$\int \Big(\prod_i \mathrm{d}\theta_i^* \mathrm{d}\theta_i\Big)\theta_k\theta_l^* e^{-\theta_i^* A_{ij}\theta_j} \tag{1}$$

  where the θ_i's are Grassmann variables.

Using the Dirac Lagrangian for free fermions: [1]

$$\mathcal{L}[\psi,\bar{\psi}] = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{2}$$

and the Gaussian integral you just evaluated:

- Calculate the fermion propagator and write the expression in the standard way (integral over Fourier space)

$$\langle\Omega|T\big\{\psi(x)\bar{\psi}(y)\big\}|\Omega\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\psi,\bar{\psi}]}\,\psi(x)\bar{\psi}(y)}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\psi,\bar{\psi}]}} = S_F(x,y) \tag{3}$$

You should find that

$$S_F(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{i(p\!\!\!/ + m)}{p^2 - m^2} e^{-ip(x-y)} \tag{4}$$

- Calculate the four-point Gaussian integrals

$$\int \Big(\prod_i \mathrm{d}\theta_i^* \mathrm{d}\theta_i\Big)\theta_k\theta_l\theta_m\theta_n e^{-\theta_i^* A_{ij}\theta_j} \tag{5a}$$

$$\int \Big(\prod_i \mathrm{d}\theta_i^* \mathrm{d}\theta_i\Big)\theta_k\theta_l\theta_m^*\theta_n^* e^{-\theta_i^* A_{ij}\theta_j} \tag{5b}$$

- Use this to calculate the four-point fermion functions

$$\langle\Omega|T\big\{\psi(x)\psi(y)\psi(z)\psi(w)\big\}|\Omega\rangle \tag{6a}$$

$$\langle\Omega|T\big\{\psi(x)\psi(y)\bar{\psi}(z)\bar{\psi}(w)\big\}|\Omega\rangle \tag{6b}$$

[1] We have made use of Feynman slash notation $a\!\!\!/ = \gamma_\mu a^\mu$

The result should come in pairs of fermion propagators. If the result has a propagator from x to y we say that the field at position x has been contracted with the field at position y. For the propagator we calculated above, this can be written as

$$\langle\Omega|\psi(x)\bar{\psi}(y)|\Omega\rangle = S_F(x,y) = \overbracket{\psi(x)\bar{\psi}}(y) \tag{7}$$

- Define the rules for $\overbracket{\psi(x)\psi}(y)$ and $\overbracket{\bar{\psi}(x)\bar{\psi}}(y)$, and write down the two four-point functions in the Wick contraction notation.

Problem 2. *Yukawa Theory*

In this exercise we will explore an interacting theory between fermions and scalars known as Yukawa Theory. This theory can work as sort of a simplified toy model[2] of QED where the photons are replaced by massive scalar fields. The Lagrangian of the theory is:

$$\mathcal{L}\big[\phi,\psi,\bar{\psi}\big] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}\big(i\partial\!\!\!/ - m_\psi\big)\psi + g\bar{\psi}\psi\phi \tag{8}$$

- Start by writing down the Feynman rules for the theory.

As with most interactive theories, we cannot solve n-point expectation values analytically, so we will have to resort to perturbation theory.

- Assume for now that the coupling constant g is a small quantity, and calculate the corrections to the fermion and boson propagators to $\mathcal{O}\big(g^2\big)$

$$\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle \;\;\&\;\; \langle\Omega|T\big\{\psi(x)\bar{\psi}(y)\big\}|\Omega\rangle \tag{9}$$

  Hint: The Wick contractions might come in handy for bookkeeping and Feynman rules for the integrals.

- Look at anisotropic n-point functions mixing bosons and fermions of the type

$$\langle\Omega|T\big\{\phi(x_1)\cdots\phi(x_n)\psi(y_1)\cdots\psi(y_m)\bar{\psi}(z_1)\cdots\bar{\psi}(z_k)|\Omega\rangle \tag{10}$$

  What is the lowest $n>0$ and $m+k>0$ that gives a nonvanishing contribution? What is the next? Draw their diagrams.

- Draw all diagrams up to $\mathcal{O}\big(g^4\big)$ that contribute to the fermionic and bosonic four-point functions and write down their respective symmetry factors:

$$\langle\Omega|T\big\{\phi(x)\phi(y)\phi(z)\phi(w)\big\}|\Omega\rangle \;\;\&\;\; \langle\Omega|T\big\{\psi(x)\bar{\psi}(y)\psi(z)\bar{\psi}(w)\big\}|\Omega\rangle \tag{11}$$

Bonus:

Introduce the $\lambda\phi^4$ interaction term to the Yukawa Lagrangian and draw all diagrams that contribute to the two propagators to $\mathcal{O}\big(\lambda g^4\big)$.

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[2] Same mathematical structure or diagrams, does however not mean that they are necessarily related