Exercise Sheet VII

June 03 [Solution June 09]

Problem 1. *SU(3) and U(3) gauge theories*

In this exercise we will look at the gauge theories we get when we couple SU(3) to the Dirac Lagrangian, and look at a couple of scattering amplitudes in this case. From the lecture we know that we can accomplish this by trading in the derivative operator with the covariant derivative (minimal coupling), and introduce the field strength tensor:

$$\mathcal{L} = \bar{\psi}\big(i(\partial\!\!\!/ - ig A\!\!\!/) - m\big)\psi - \frac{1}{4}\,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} \tag{1}$$

Here the fields ψ have an additional index, the group index, and the A's are $D_R \times D_R$ matrices, where D_R is the dimension of the representation.

We will stick with SU(3) in the fundamental representation for now. This means that the fermions will have an additional 3-dimensional vector, specifying their colour (red, blue or green), and there will be 8 generators which make up the gluons. We will use the Gell-Mann matrices as the basis to form the generators T_α. For our generators to satisfy the trace identity, $\mathrm{tr}\, T_\alpha T_\beta = \frac{1}{2}\delta_{\alpha\beta}$, they need to be normalised to $T_\alpha = \frac{1}{2}\lambda_\alpha$, where λ are the Gell-Mann matrices:

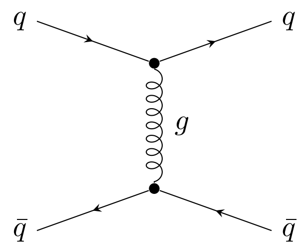
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

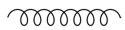
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Write down all the fermion related Feynman rules for this Lagrangian.
- *(Bonus)* Write out the field strength tensor and verify the three-point and four-point gauge self-interaction rules.

Next we will calculate quark-anti-quark scattering with gluons. The Feynman diagram for which is



We will fix the gauge to be Feynman gauge, which means that the gluon propagator will be

 $= \dfrac{-ig_{\mu\nu}\delta_{ab}}{p^2}$ (2)

- Calculate the scattering amplitude $|\mathcal{M}|^2$ for this process, averaging over spin, but leaving the colour unfixed for now.
- Compare the result to that of standard scattering with photons, what is different?

There are two distinct ways one can fix the colour indices of the above diagram. Either picking from the colour octet:

$$
\begin{aligned}
&(r\bar{b} + b\bar{r})/\sqrt{2}, && -i(r\bar{b} - b\bar{r})/\sqrt{2}, \\
&(r\bar{r} - b\bar{b})/\sqrt{2}, && (r\bar{g} + g\bar{r})/\sqrt{2}, \\
&-i(r\bar{g} - g\bar{r})/\sqrt{2}, && (b\bar{g} + g\bar{b})/\sqrt{2}, \\
&-i(b\bar{g} - g\bar{b})/\sqrt{2}, && (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}
\end{aligned}
$$

or from the colour singlet:

$$(r\bar{r} + b\bar{b} + g\bar{g})/\sqrt{3}$$

- Calculate the scattering angle with the quark-anti-quark pair in the colour octet.
- Redo the calculation with a colour singlet pair. What is different? What does this imply?

Finally we will "upgrade" our theory to be invariant under local U(3) transformations instead.

- How does this change affect the fermionic fields?
- Find the missing generator and normalise it so it follows the trace identity.
- How does this generator transform under SU(3)? Comments?
- Redo the scattering calculation using the "new" gluon. How does it interact with the colour charges? Which particle does this gluon look like? Could it act as a replacement?