Exercise sheet X

June 24 [solution: June 30]

Problem 1 [*Feynman parametrization*] Many computations in perturbation theory require the use of the so-called *Feynman parametrization.* It relies on the introduction of an additional *Feynman parameter* that replaces a product of denominators with an integral over this parameter. In the simplest case:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}. \tag{1}$$

Prove this formula. Then, prove the general case by induction:

$$\frac{1}{A_1 \ldots A_n} = (n-1)! \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_n \frac{\delta(x_1 + \ldots + x_n - 1)}{(x_1 A_1 + \ldots + x_n A_n)^n}. \tag{2}$$

Hint: derive first the following case:

$$\frac{1}{AB^n} = \int_0^1 dx \int_0^1 dy\, \delta(x + y - 1) \frac{n y^{n-1}}{(xA + yB)^{n+1}} \tag{3}$$

by taking the $n-1$-th derivative of (1). After that, use this result to rewrite $1/(x_1 A_1 + \ldots)^n \cdot 1/A_{n+1}$ that you get in induction by assuming the validity of (2). Finally, make a change of variables: $z_k = y x_k$ $(k = 1, \ldots, n)$, $z_{n+1} = 1 - y$ and use the property of the δ-function: $\delta(ax) = \delta(x)/|a|$.

Problem 2 [*Dimensional regularization*] Regularize the following integral, needed for the renormalization of the coupling in the ϕ^4 theory in Euclidean spacetime at $\mathcal{O}(\lambda^2)$:

$$I(p_i, p_j) = \left(\mu^{4-d}\right)^2 \int d^d k \frac{1}{k^2 + m^2} \frac{1}{(k + p_i + p_j)^2 + m^2}. \tag{4}$$

To proceed, you need to use the Feynman parametrization trick from the previous exercise. You should get the following result:

$$I(p_i, p_j) = \mu^\epsilon \pi^2 \left(\frac{2}{\epsilon} - \gamma_E - F((p_i + p_j)^2, m, \mu) + \mathcal{O}(\epsilon) \right), \tag{5}$$

where $\epsilon = 4 - d$, $F((p_i + p_j)^2, m, \mu) = \int_0^1 dx \ln \left(\frac{\pi\left((p_i+p_j)^2 x(1-x) + m^2\right)}{\mu^2} \right)$.

Problem 3 [*Renormalized coupling*] The proper function $\Gamma_4(x_1, \ldots, x_4)$ is:

$$\begin{aligned} \Gamma_4(x_1, \ldots, x_4) \quad = \quad & \frac{1}{(2\pi)^d} \int d^d p_1 e^{-p_1 x_1} \ldots \frac{1}{(2\pi)^d} \int d^d p_4 e^{-p_4 x_4}\ \delta(p_1 + \ldots + p_4) \\ & \left(\mu^{4-d} \lambda (2\pi)^d - \frac{\lambda^2}{2} \left(I(p_1, p_4) + I(p_2, p_4) + I(p_3, p_4) \right) + \mathcal{O}(\lambda^3) \right). \end{aligned} \tag{6}$$

Using the result from the previous exercise, show that its Fourier transform can be written as:

$$\begin{aligned}\tilde{\Gamma}_4(p_1,\ldots,p_4) &= (2\pi)^4\mu^\epsilon\lambda\delta(p_1+\ldots+p_4)\Big[1-\frac{3\lambda}{32\pi^2}\Big(\frac{2}{\epsilon}-\gamma_E-F((p_1+p_4)^2,m,\mu)+\\ &- F((p_2+p_4)^2,m,\mu)-F((p_3+p_4)^2,m,\mu)+\mathcal{O}(\epsilon)\Big)+\mathcal{O}(\lambda^2)\Big] = \\ &= (2\pi)^4\mu^\epsilon\lambda\delta(p_1+\ldots+p_4)\left[1-\frac{3\lambda}{16\pi^2\epsilon}+\text{ finite }+\mathcal{O}(\lambda^2)\right].\end{aligned} \tag{7}$$

Impose the renormalization condition (or choose some other one):

$$\tilde{\Gamma}_4(p_1,\ldots,p_4)\Big|_{p_1=\ldots=p_4=0} = (2\pi)^4\mu^\epsilon\delta(p_1+\ldots+p_4)\lambda_R\Big|_{p_1=\ldots=p_4=0} \tag{8}$$

and show that the bare coupling is infinite, e.g. for the above renormalization condition:

$$\lambda = \lambda_R\left(1+\frac{3\lambda_R}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma_E-F(0,m,\mu)\right)+\mathcal{O}(\lambda_R^2)\right). \tag{9}$$

Hence, perturbation theory makes sense only in the renormalized coupling λ_R.