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Bachelor Thesis

Heavy Quark Mass and Spin Effects in Bottomonium Bound States

by

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Contents

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Abstract

In this thesis, the energy levels of different bottomonium bounded states were computed numerically within the confines of different corrections stemming from the static quark potential derived from non-relativistic QCD as presented in [2].
These different corrections were then compared with each other as well as with previously obtained results from Michael Eichberg and Marc Wagner in [11].
Furthermore, a method was introduced for obtaining the asymptotic solutions of a differential equation with poles around zero.

Introduction

The bottom quark is the second heaviest quark and the heaviest expected to pair with its antiparticle to form a flavorless meson, in this case bottomonium. The only heavier quark, i.e. the top quark, decays so rapidly that the associated quarkonium, toponium also referred to as the theta meson, is not presumed to exist. The first state of bottomonium to be discovered was $\Upsilon(1S)$ with a total angular momentum of 1 and a parity as well as a charge parity of -1 in Fermilab in 1977. Since then, many more states of bottomonium have been observed in nature, or more precisely, in colliders. This has led to a plethora of experimental values, making bottomonium ideal for testing new developments in non-perturbative approximations in QCD especially since its properties as a meson can only be approached from a non-perturbative standpoint. Additionally, the large mass of bottomonium allows for a non-relativistic consideration and thus facilitates its use to gauge the potential of developed non-relativistic QCD models. Non-relativistic QCD, abbreviated NRQCD, is an effective field theory derived from regular QCD. More precisely, it is obtained when integrating out energy scales above the mass m while a further simplification, the so-called potential NRQCD and abbreviated pNRQCD, further integrates out all energy scales above mv. See [3] and the introduction of [7] for a more detailed review.

On account of these theories, Antonio Pineda and Antonio Vairo derived the full static quark potentials up to order $1/m^2$ in [2]. In this thesis, their result will be used to compute energy levels of different states of bottomonium. This is accomplished by appealing to hte well established method of solving the Schrödinger equation associated with this potential. Here two main approaches come to mind. On the one hand, a perturbative approach is possible. This was pursued by Michael Eichberg and Marc Wagner in [11]. In this thesis, however, another approach will be taken. Instead of considering parts of the static quark potential as perturbations of some main part, the entire static quark potential with its associated Schrödinger equation will be considered. This Schrödinger equation is a differential equation which is numerically solvable. In order to accomplish this, the first chapter is devoted to introducing the static quark potential and to conducting some preliminary simplifications such as for example locating a suitable spin basis. The second chapter will then present the full derivation of the differential equation, corresponding to the problem at hand. In order to solve this differential equation numerically, the asymptotic behavior of the solution will be required which will be addressed in the third chapter. Here a wide variety of different techniques will be necessary to arrive at a satisfactory description of the asymptotic behavior. At last, it is possible to numerically compute the energy levels of different states of bottomonium. As a conclusion, these differ by around 0.8% from the experimental values.

Convention: Throughout this thesis, natural units with $\hbar = 1$ and $c = 1$ will be used.

Notation:

Symbol	Definition	Description
$[A, B]_-$	$AB - BA$	Commutator
$[A, B]_+$	$AB + BA$	Anticommutator
$\mathbf{e}_\mathbf{r}$	$\mathbf{r}/\lvert\mathbf{r}\rvert$	Normalization of the vector $\mathbf{r}$

Chapter 1

Background

The main goal of this chapter is to introduce the static quark potential which forms the backbone of the Schrödinger equation, eventually leading to the energy values of the different states of bottomonium. After having introduced the required potential, two other instrumental principles, the singlet and triplet states of two-particle systems and angular momentum coupling with the accompanying Clebsch-Gordon coefficients, will be revisited. Together this will yield a very convenient basis of the Hilbert space. This basis will be used in the subsequent chapter to both simplify the Schrödinger equation accompanying the aforementioned potential as well as provide an insight in the reasoning behind the upcoming simplifications.

1.1 Potentials

In 2000, Antonio Pineda and Antonio Vairo (see [2]) deduced the following static quark potential in its present form which governs the behavior of a heavy quark-antiquark pair. In their work, the partaking quark and antiquark are not required to possess the same mass. In this thesis, only bottomonium ($b\overline{b}$) will be considered, and thus the quark and antiquark will always have the same mass. This simplifies the potential leading to the following form:

Static Quark Potentials (see [2])

Consider a (heavy) quark Q of mass m together with its antiquark $\overline{Q}$ such that their center of mass lies at the origin. Assume the following notational conventions:

	Q	$\overline{Q}$	$Q\overline{Q}$
position	$\mathbf{r}/2$	$-\mathbf{r}/2$	
momentum			$\mathbf{p}$
spin	$\mathbf{S}_1$	$\mathbf{S}_2$	$\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$
angular momentum			$\mathbf{L}$
total angular momentum			$\mathbf{J}=\mathbf{S}+\mathbf{L}$

The effective potential $V(\mathbf{r},\mathbf{p},\mathbf{L},\mathbf{S}_1,\mathbf{S}_2,\mathbf{S})$ resulting from the strong interaction between a quark and its antiquark spaced $\mathbf{r}$ apart can be written as

$$V(\mathbf{r},\mathbf{p},\mathbf{L},\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}) = V^{(0)}(\mathbf{r}) + \frac{1}{m}V^{(1)}(\mathbf{r}) + \frac{1}{m^2}V^{(2)}(\mathbf{r},\mathbf{p},\mathbf{L},\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}) + \mathcal{O}\left(\frac{1}{m^3}\right)$$

when expanded in the quark mass m. Here

$$V^{(0)}(r) = \frac{-e}{r} + \sigma r, \qquad (0^{\text{th}}\text{ Order})$$

$$V^{(1)}(r) = \frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r), \qquad (1^{\text{st}}\text{ Order})$$

$$V^{(2)}(\mathbf{r},\mathbf{p},\mathbf{L},\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}) = V_{\text{SD}}(\mathbf{r},\mathbf{L},\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}) + V_{\text{SI}}(\mathbf{r},\mathbf{p},\mathbf{L}). \qquad (2^{\text{nd}}\text{ Order})$$

The **s**pin **d**ependent part $V_{\mathrm{SD}}(\mathbf{r}, \mathbf{L}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S})$ of the second-order correction is

$$V_{\mathrm{SD}}(\mathbf{r}, \mathbf{L}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}) = V_{\mathrm{LS}}(r)\mathbf{LS} + V_{S_{12}}(r)\mathbf{S}_{12}(\mathbf{r}) + V_{S^2}(r)(\mathbf{S}_1\mathbf{S}_2)$$

with the mixed spin term $\mathbf{S}_{12}(\mathbf{r}) = (\mathbf{S}_1\mathbf{e}_\mathbf{r})(\mathbf{S}_2\mathbf{e}_\mathbf{r}) - \frac{\mathbf{S}_1\mathbf{S}_2}{3}$.
The **s**pin **i**ndependent part $V_{\mathrm{SI}}(\mathbf{r}, \mathbf{p}, \mathbf{L})$ of the second-order correction is

$$V_{\mathrm{SI}}(\mathbf{r}, \mathbf{p}, L) = V_{\mathrm{r}}(r) + 2\mathbf{L}^2 V_{\mathrm{L}}(r) + [\mathbf{p}^2, V_{\mathrm{p}}(r)]_+.$$

It is important to note that the static quark potential shown above is only accurate up to some unknown additive offset. This is not relevant for the concluding discussions since such an offset can easily be deduced by matching a theoretically computed energy state with its experimental analogon.

Although the constants as well as the potentials $V_{\mathrm{LS}}, V_{S_{12}}, V_{S^2}, V_{\mathrm{r}}(r), V_{\mathrm{L}}(r)$, and $V_{\mathrm{p}}(r)$ do not play an immediate role, it is fitting to write out their full forms. See Appendix A for a derivation.

Explicit Forms for the Radial Potentials (due to [1], [6], [10])

$$V_{\mathrm{LS}}(r) = \frac{5e}{2r^3} + \frac{\sigma}{2r} \qquad V_{\mathrm{r}}(r) = \left(\frac{3}{4} + d_s\right)\frac{\delta(r)}{r^2}$$

$$V_{S_{12}}(r) = \frac{3e}{r^3} \qquad V_{\mathrm{L}}(r) = \frac{e}{4r^3} - \frac{\sigma}{12r}$$

$$V_{S^2}(r) = \frac{2e\delta(r)}{r^2} \qquad V_{\mathrm{p}}(r) = -\frac{C_F\alpha_s\mu}{2\pi} - \frac{e}{2r}$$

The potentials $V_{S^2}(r)$ and $V_{\mathrm{r}}(r)$ are only shown for completeness. They will not be used throughout this thesis.

$$m_b = 4.977 \text{ GeV} \qquad \sigma = 0.282159 \text{ GeV}^2$$

$$\alpha_s = 0.2815 \qquad \mu = 1.5879557\frac{\pi}{a} \qquad \text{(see [1, p. 37])}$$

$$C_F = 4/3 \qquad e = C_F\alpha_s \qquad \text{(see [1, p. 97])}$$

The string tension σ was determined by comparing certain computed energies with their experimental counterparts. The details of this will be discussed in chapter 4.

At this point, it is appropriate to discuss the zero order term of the static quark potential. This term is ubiquitous in the context of quark confinement and bears the name Cornell potential. Recall that the strong force increases in magnitude with an increase of distance of the participating particles. This is similar to a string being pulled taut. It turns out that this behavior is approximately linear for large distances and again similar to the tension of string. In accordance with this analagon, the proportionality constant σ is also referred to as the **string tension**. This linear behavior can further be deduced from the zero order term $-e/r + \sigma r$ of the static quark potential. A closer analysis of the behavior of the strong interaction at short distances allows for the recreation of the $1/r$ term. The term string breaking now stems from the fact that as the distance between the quark-antiquark pair increases, so does the required energy to keep it at this distance. At some point, the energy will suffice to create another quark-antiquark pair which interacts with the already present one to form two new mesons leading to a more favorable state. Of course, if string breaking is considered, the associated potential is more complicated than the simple linear relationship mentioned before. In this thesis, however, string breaking will not be examined, though perhaps it will be returned to in the future. Instead only string tension will play a role.

The first order term is in the same spirit of the zeorth order one and simply expands on it by including higher order terms. It should be noted that up until this point, the spin of the quark and antiquark are of no relevance. This changes with the second order term which neatly separates into a spin dependent and an independent part. It is now the main task of this thesis to study and gauge the influence of this term. In order to do this, the concept of a singlet and triplet state will be useful which will make up the contents of the next section.

1.2 Hilbert Space and Basis of Spin Component

This section will review the standard methodology of incorperating spin dynamics into otherwise normal Hilbert space and thus giving the differential equation of the previous section firm footing inside an augmented Hilbert space.

In the conventional situation of a one-particle fermonic system, it is possible to artificially introduce a spin nature by augmenting the conventional (no spin) Hilbert space. For simplicity, assume further that the fermion has a spin of $1/2$. Let the conventional Hilbert space without spin effects be denoted by $\mathcal{H}$ and add to it the two possible spin configurations $|\uparrow\rangle$ and $|\downarrow\rangle$. This is accomplished by considering the Hilbert space

$$\mathcal{H}_S := \mathcal{H} \otimes_{\mathbb{C}} (\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle) \simeq \mathcal{H}|\uparrow\rangle + \mathcal{H}|\downarrow\rangle \simeq \mathcal{H}^2.$$

Following convention $|\uparrow\rangle$ and $|\downarrow\rangle$ should be the eigenstates of the spin operator in the $z-$direction with an eigenvalue $\pm 1/2$, respectively. Interpreting the first component of $\mathcal{H}^2$ as the spin-up component and the second as the spin-down component, the spin operators take on the following form

$$\mathbf{S}_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{S}_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathbf{S}_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

including the Pauli matrices σ_x, σ_y, and σ_z. As expected, the (squared) magnitude of the spin is

$$\mathbf{S}^2 = \frac{1}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{4}\mathbb{1} = \frac{1}{2}\left(\frac{1}{2}+1\right)\mathbb{1}.$$

To introduce a spin dynamic into a two particle system, it now suffices to „glue" (tensor) the Hilbert spaces of the two one-particle systems with spin effects together. In the present case of a quark and its antiquark, the partaking particles are non-identical, and thus the spin statistics theorem does not weigh in, and furthermore the one-particle Hilbert spaces of the quark and antiquark do not differ. All in all, while denoting the individual one-particle Hilbert space $\mathcal{H}_S$ with spin and $\mathcal{H}$ without spin, the Hilbert space of the two particle system can be chosen as the tensor product of the one-particle Hilbert spaces (with spin)

$$\mathcal{H}^{(2)} = \mathcal{H}_S \otimes_{\mathbb{C}} \mathcal{H}_S \simeq \mathcal{H}|\uparrow\rangle \otimes |\uparrow\rangle + \mathcal{H}|\uparrow\rangle \otimes |\downarrow\rangle + \mathcal{H}|\downarrow\rangle \otimes |\uparrow\rangle + \mathcal{H}|\downarrow\rangle \otimes |\downarrow\rangle \simeq \mathcal{H}^4.$$

In accordance with this definition, the spin operators of the individual particles are

$$\mathbf{S}_1 := \mathbf{S} \otimes \mathrm{id} : \mathcal{H}^{(2)} \to \mathcal{H}^{(2)} \qquad \text{and} \qquad \mathbf{S}_2 := \mathrm{id} \otimes \mathbf{S} : \mathcal{H}^{(2)} \to \mathcal{H}^{(2)},$$

respectively. This allows for the definition of the total spin as $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = \mathbf{S} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbf{S}$.

One choice of a spin basis is now immediately evident, namely

$$|\uparrow\rangle \otimes |\uparrow\rangle, \qquad |\uparrow\rangle \otimes |\downarrow\rangle, \qquad |\downarrow\rangle \otimes |\uparrow\rangle, \qquad \text{and} \qquad |\downarrow\rangle \otimes |\downarrow\rangle.$$

The issue with this choice of basis is, however, that it is only an eigenstate of $\mathbf{S}_1$ and $\mathbf{S}_2$ but not of the total spin $\mathbf{S}$ which occurs predominately in the static quark potential as will become more obvious in the next chapter.

To address this, consider the so-called singlet and triplet basis consisting of a singlet state and of three triplet states. The spin configuration

$$\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}$$

is the **singlet** state and is also a simultaneous eigenstate of $\mathbf{S}_z$ and $\mathbf{S}^2$ with

$$\mathbf{S}_z\left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right) = 0 \cdot \left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right), \qquad \mathbf{S}^2\left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right) = 0 \cdot \left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right).$$

The remaining spin configurations

$$|\uparrow\rangle \otimes |\uparrow\rangle, \qquad |\downarrow\rangle \otimes |\downarrow\rangle, \qquad \text{and} \qquad \frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}$$

are called the **triplet** states and are simultaneous eigenstates of $\mathbf{S}_z$ and $\mathbf{S}^2$ with

$$\begin{aligned} \mathbf{S}_z(|\uparrow\rangle \otimes |\uparrow\rangle) &= |\uparrow\rangle \otimes |\uparrow\rangle, & \mathbf{S}^2(|\uparrow\rangle \otimes |\uparrow\rangle) &= 2|\uparrow\rangle \otimes |\uparrow\rangle, \\ \mathbf{S}_z(|\downarrow\rangle \otimes |\downarrow\rangle) &= -|\downarrow\rangle \otimes |\downarrow\rangle, & \mathbf{S}^2(|\downarrow\rangle \otimes |\downarrow\rangle) &= 2|\downarrow\rangle \otimes |\downarrow\rangle, \\ \mathbf{S}_z\left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right) &= 0 \cdot \left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right), & \mathbf{S}^2\left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right) &= 2\left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right). \end{aligned}$$

It is now possible to write any element $\psi \in \mathcal{H}^{(2)}$ in four components as

$$\psi = \begin{pmatrix} \psi_{|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\downarrow\rangle\otimes|\downarrow\rangle} \end{pmatrix}$$

$$= \psi_{|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle} \left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}} \right) + \psi_{|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle} \left(\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}} \right)$$
$$+ \psi_{|\uparrow\rangle\otimes|\uparrow\rangle} |\uparrow\rangle \otimes |\uparrow\rangle + \psi_{|\downarrow\rangle\otimes|\downarrow\rangle} |\downarrow\rangle \otimes |\downarrow\rangle.$$

Now, $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}$ can be defined in this basis for the spin component of the elements in $\mathcal{H}^{(2)}$.

Spin Operators in Basis of Singlet and Triplet States

The spins of the individual particles take on the following form if expressed in terms of the singlet and triplet basis:

$$\mathbf{S}_{1,x} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{pmatrix} \qquad \mathbf{S}_{2,x} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

$$\mathbf{S}_{1,y} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} & 0 & 0 \\ i/\sqrt{2} & i/\sqrt{2} & 0 & 0 \end{pmatrix} \qquad \mathbf{S}_{2,y} = \frac{1}{2} \begin{pmatrix} 0 & 0 & i/\sqrt{2} & i/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & -i/\sqrt{2} & 0 & 0 \\ -i/\sqrt{2} & i/\sqrt{2} & 0 & 0 \end{pmatrix}$$

$$\mathbf{S}_{1,z} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{S}_{2,z} = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then the combined spin of both particles can be written as

$$\mathbf{S}_x = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \qquad \mathbf{S}_y = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & -i\sqrt{2} \\ 0 & -i\sqrt{2} & 0 & 0 \\ 0 & i\sqrt{2} & 0 & 0 \end{pmatrix} \qquad \mathbf{S}_z = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Furthermore, $\mathbf{S}^2$ is

$$\mathbf{S}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

As expected and required, $\mathbf{S}_z$ and $\mathbf{S}^2$ are both diagonal.

1.3 Basis of Momentum Component

In the present problem of a quark-antiquark pair, there are two angular momenta, namely the spin and the conventional angular momentum. For reasons that will quickly become apparent in the following chapter, it is of paramount importance to locate eigenstates of the total momenta operators $\mathbf{J}^2$ and $\mathbf{J}_z$. This is accomplished by using the Clebsch-Gordon coefficients which are quickly reviewed in this section.

The total angular momentum operator in the present situation is composed of the total spin of both particles and of the combined conventional momentum. Since both the quark and antiquark can only possess a spin of $1/2$, the total spin can only be 0 or 1 with the z-coordinate behaving accordingly. The range of the total conventional momentum knows no bounds, however.

On a very general level, consider two angular momenta operators $\mathbf{L}_1$ and $\mathbf{L}_2$ which commute together with their sum $\mathbf{J} = \mathbf{L}_1 + \mathbf{L}_2$. It is possible to express the sought after simultaneous eigenstates

$$|j, j_z; l_1; l_2\rangle = \sum_{\substack{|l_1 - l_2| \leq j \leq |l_1 + l_2| \\ l_{1,z} + l_{2,z} = j_z}} \Gamma_{j,j_z;l_1,l_{1,z};l_2,l_{2,z}} |l_1, l_{1,z}; l_2, l_{2,z}\rangle$$

of $\mathbf{J}^2, \mathbf{J}_z, \mathbf{L}_1^2$, and $\mathbf{L}_2^2$ in terms of the simultaneous eigenstates $|l_1, l_{1,z}; l_2, l_{2,z}\rangle$ of $\mathbf{L}_1, \mathbf{L}_{1,z}, \mathbf{L}_2$, and $\mathbf{L}_{2,z}$. The occurring coefficients $\Gamma_{j,j_z;l_1,l_{1,z};l_2,l_{2,z}}$ are called the **Clebsch-Gordan coefficients** and are *independent* of the precise implementation of $\mathbf{J}, \mathbf{L}_1$, and $\mathbf{L}_2$.

Now, it is possible to specialize the very general above form to the present specific situation with $\mathbf{L}_1 = \mathbf{S}$ and $\mathbf{L}_2 = \mathbf{L}$. For this, it is essential to determine the simultaneous eigenstates of $\mathbf{L}^2$ and $\mathbf{L}_z$ as well as $\mathbf{S}^2$ and $\mathbf{S}_z$. Returning to the explicit construction of the underlying Hilbert space

$$\mathcal{H}^{(2)} = (\mathcal{H}|\uparrow\rangle \otimes |\uparrow\rangle) \oplus (\mathcal{H}|\downarrow\rangle \otimes |\downarrow\rangle) \oplus \left(\mathcal{H}\frac{|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right) \oplus \left(\mathcal{H}\frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}\right),$$

observe that the spin part is captured in the singlet and triplet states while the conventional angular momentum part resides solely in $\mathcal{H}$. Therefore, it is possible to locate their respective eigenstates individually before combining them to the final result.

The eigenstates of the conventional momenta are the spherical harmonicals $Y_l^m(\theta, \varphi)$ with

$$\mathbf{L}^2 Y_l^m(\theta,\varphi) = l(l+1)Y_l^m(\theta,\varphi) \qquad \text{and} \qquad \mathbf{L}_z Y_l^m(\theta,\varphi) = mY_l^m(\theta,\varphi).$$

Do note, however, that $\mathbf{L}^2$ and $\mathbf{L}_z$ only effect the θ and φ part of an element of the Hilbert space.

The eigenstates of the spin operator $\mathbf{S}^2$ and $\mathbf{S}_z$ are immediately clear due to the definition of the singlet and triplet basis

$$S^2\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0(0+1)\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad S_z\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

$$S^2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 1(1+1)\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad S_z\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad S^2\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 1(1+1)\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad S_z\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 1\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad S^2\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 1(1+1)\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad S_z\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = -1\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Combining these eigenstates leads to the following expression for the simultaneous eigenstates

$$|0,0;l,m\rangle = \begin{pmatrix} Y_l^m(\theta,\varphi)\\0\\0\\0\end{pmatrix},$$

$$|1,-1;l,m\rangle = \begin{pmatrix}0\\0\\0\\Y_l^m(\theta,\varphi)\end{pmatrix}, \qquad |1,0;l,m\rangle = \begin{pmatrix}0\\Y_l^m(\theta,\varphi)\\0\\0\end{pmatrix}, \qquad |1,1;l,m\rangle = \begin{pmatrix}0\\0\\Y_l^m(\theta,\varphi)\\0\end{pmatrix}$$

of $\mathbf{L}^2, \mathbf{L}_z, \mathbf{S}^2$, and $\mathbf{S}_z$. The resulting states $|s, s_z; l, m\rangle$ then satisfy

$$\mathbf{S}^2|s,s_z;l,m\rangle = s(s+1)|s,s_z;l,m\rangle, \qquad \mathbf{S}_z|s,s_z;l,m\rangle = s_z|s,s_z;l,m\rangle,$$
$$\mathbf{L}^2|s,s_z;l,m\rangle = l(l+1)|s,s_z;l,m\rangle, \qquad \mathbf{L}_z|s,s_z;l,m\rangle = m|s,s_z;l,m\rangle.$$

Using the Clebsch-Gordon coefficients, it is now possible to write down the simultaneous eigenstates of $\mathbf{J}^2, \mathbf{J}_z, \mathbf{S}^2$, and $\mathbf{L}^2$. As an example, consider

$$\begin{aligned}|1,0;1;2\rangle &= \Gamma_{1,0;1,1;2,-1}|1,1;2,-1\rangle + \Gamma_{1,0;1,0;2,0}|1,0;2,0\rangle + \Gamma_{1,0;1,-1;2,1}|1,-1;2,1\rangle\\ &= \sqrt{\frac{3}{10}}|1,1;2,-1\rangle - \sqrt{\frac{2}{5}}|1,0;2,0\rangle + \sqrt{\frac{3}{10}}|1,-1;2,1\rangle\\ &= \left(0, -\sqrt{\frac{2}{5}}Y_2^0(\theta,\varphi), \sqrt{\frac{3}{10}}Y_2^{-1}(\theta,\varphi), \sqrt{\frac{3}{10}}Y_2^1(\theta,\varphi)\right).\end{aligned}$$

In the next chapter, this basis of the Hilbert space will be used to dramatically simplify the Schrödinger equation.

Chapter 2

Differential Equations

In the previous chapter, the static quark potential $V(\mathbf{r}, \mathbf{p}, \mathbf{L}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S})$ was introduced. This naturally leads to a Hamilton operator

$$\mathbf{H} = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r}, \mathbf{p}, \mathbf{L}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}),$$

where $\mu = m/2$ denotes the reduced mass of the two-particle system. The main aim of this thesis is to solve the differential equations

$$\mathbf{H}|\psi\rangle = \mathbf{H} \begin{pmatrix} \psi_{|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\downarrow\rangle\otimes|\downarrow\rangle} \end{pmatrix} = E \begin{pmatrix} \psi_{|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\uparrow\rangle\otimes|\uparrow\rangle} \\ \psi_{|\downarrow\rangle\otimes|\downarrow\rangle} \end{pmatrix} = E|\psi\rangle$$

for E to determine the energies of the different bottomonium states.

It is imperative for this chapter to simplify the above equation as far as possible to lessen the load on the numerical calculation. More specifically, all dependencies on φ and θ in terms of spherical coordinates will be eliminated. This simplification process is undertaken in three steps. First, $|\psi\rangle$ will be expressed in terms of the basis found at the end of the previous chapter. This results in the φ and θ part of the wave function residing entirely in the basis elements $|j, j_z; s; l\rangle$ while the entire dependence on the radius r will remain in the coefficients. Secondly, the effect of $\mathbf{H}$ on the basis elements $|j, j_z, s, l\rangle$ is studied. Here (and also in the first step) it will be explained how knowledge regarding the parity and C (charge) parity of the partaking meson can be used to reduce the complexity of the problem further. Lastly, $\mathbf{H}$ will be applied to $|\psi\rangle$ and then decomposed again into the states $|j, j_z; s; l\rangle$ after which equating coefficients with $E|\psi\rangle$ yields the sought after differential equations.

2.1 Reexpressing $|\psi\rangle$

The first task at hand is to determine a closed form for the parity and the C parity of the states $|j, j_z; s; l\rangle$. Sadly, this is not as simple as considering the sign change under the $\mathbf{r} \mapsto -\mathbf{r}$ (for regular parity). This is due to the fact that quarks possess a certain intrinsic parity which needs to be taken into account. For the sake of completeness the full derivation is repeated here:

First, consider the regular parity of $|j, j_z; s; l\rangle$. Since this state is entirely composed of spherical harmonics associated to a conventional angular momentum of l, the parity of the state coincides with that of the aforementioned spherical harmonic, i.e. $(-1)^l$. Now, all that remains is to consider the intrinsic parity of the system. This is the product of the intrinsic parity of the quark and its antiquark. Using the Dirac equation, it is possible to compute the intrinsic parity of the quark as 1 and that of the antiquark as -1 (see [8, Chapter 3.8]). Taking the product of all these "parities" results in a total parity of

$$(-1)^l \cdot 1 \cdot (-1) = (-1)^{l+1}.$$

Secondly, consider the C parity. Since charge conjugation transforms a particle into its antiparticle, the quark and antiquark swap places leading to a baseline factor of $(-1)^{l+1}$, entirely analogously to the case of regular parity. Contrary to the case of regular parity, the quark is also transformed into an antiquark and vice versa. This effectively means that in the singlet and triplet states the two particles are exchanged. Now, recall

that the singlet state ($S = 0$) is antisymmetrical, leading to a factor of (-1) while the triplet states ($S = 1$) are symmetrical, thus resulting in a factor of 1. In total, the spin part contributes a factor of $(-1)^{s+1}$ under charge conjugation. Combining these two factors yields a total C parity of

$$(-1)^{l+1}(-1)^{s+1} = (-1)^{l+s}.$$

$P-$ and $C-$parity of $|j, j_z; s; l\rangle$

The state $|j, j_z; s; p\rangle$ has a parity of $(-1)^{l+1}$ and a C parity of $(-1)^{l+s}$.

If the state $|\psi\rangle$ describes a particle with a total angular momentum of j as well as a parity and C parity of P and C, respectively, then the state can be written as

$$|\psi\rangle = \sum_{j_z=-J}^{J} \sum_{\substack{|s-l|\leq j\leq |s+l| \\ (-1)^{l+1}=P \\ (-1)^{l+s}=C}} A_{j,j_z;s;l}(r)|j, j_z; s; l\rangle. \tag{2.1}$$

Do notice the dependence of the coefficients on the magnitude r, i.e. the distance between the quark and the antiquark.

2.2 An Analysis of the Hamiltonian

The task of this section is the simplification of the Hamiltonian and the analysis of its effect on $|j, j_z; s; l\rangle$. This is accomplished by separating the Hamiltonian into two parts $\mathbf{H} = \mathbf{H}_{\text{D}} + \mathbf{H}_{\text{ND}}$ where the $|j, j_z; s; l\rangle$ are eigenstates of H_D and the dynamic, i.e. the more complicated behavior, of the Hamiltonian resides in H_{ND}.

Writing out the Hamiltonian in its fullest amounts

$$\begin{aligned}\mathbf{H} =& \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r}, \mathbf{p}, \mathbf{L}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}) \\ =& \frac{\mathbf{p}^2}{2\mu} + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) \\ &+ \frac{1}{m^2}\left(V_r(r) + 2\mathbf{L}^2 V_L(r) + [\mathbf{p}^2, V_p(r)]_+ + V_{\text{LS}}(r)\mathbf{LS} + V_{S_{12}}(r)\mathbf{S}_{12}(\mathbf{r}) + V_{S^2}(r)(\mathbf{S}_1\mathbf{S}_2)\right).\end{aligned}$$

Here the expansion of V is capped at the second order.

Recall that the main goal of this chapter is to isolate the dependence of $|\psi\rangle$ on φ and θ. Therefore, it will be useful to write $\mathbf{p}^2 = \mathbf{p}_r^2 + \frac{\mathbf{L}^2}{r^2}$ with the radial momentum $\mathbf{p}_r^2 = -\partial_r^2 - \frac{2}{r}\partial_r$, μ being the reduced mass $m/2$, and lastly the product $\mathbf{LS}$ can be reexpressed as $(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)/2$. All these considerations lead to the Hamiltonian

$$\begin{aligned}\mathbf{H} =& \frac{1}{m}\left(\mathbf{p}_r^2 + \frac{\mathbf{L}^2}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) \\ &+ \frac{1}{m^2}\left(V_r(r) + 2\mathbf{L}^2\left(V_L(r) + \frac{V_p(r)}{r^2}\right) + \left[\mathbf{p}_r^2, V_p(r)\right]_+ + V_{\text{LS}}(r)\frac{\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2}{2} + V_{S_{12}}(r)\mathbf{S}_{12}(\mathbf{r}) + V_{S^2}(r)(\mathbf{S}_1\mathbf{S}_2)\right).\end{aligned}$$

Now, three terms remain to be simplified further. Two, the anti-commutator

$$\left[\mathbf{p}_r^2, V_p(r)\right]_+ = (\mathbf{p}_r^2 V_p(r)) - 2(\partial_r V_p(r))\partial_r + 2V_p(r)\mathbf{p}_r^2$$

together with

$$\mathbf{S}_1\mathbf{S}_2 = \frac{\mathcal{DT}}{4} = \frac{\text{diag}(-3, 1, 1, 1)}{4},$$

where $\mathcal{DT}$ (**d**iagonal **t**erm) is the diagonal matrix $\mathcal{DT}$ with the diagonal entries $-3, 1, 1$, and 1, which are fairly straightforward, while the last,

$$S_{12}(\mathbf{r}) = (\mathbf{S}_1\mathbf{r})(\mathbf{S}_2\mathbf{r}) - \frac{\mathbf{S}_1\mathbf{S}_2}{3}$$

$$= \frac{r^2\sqrt{2\pi}}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2}{5}}Y_2^0(\theta,\varphi) & -\sqrt{\frac{3}{10}}Y_2^1(\theta,\varphi) & -\sqrt{\frac{3}{10}}Y_2^{-1}(\theta,\varphi) \\ 0 & \sqrt{\frac{3}{10}}Y_2^{-1}(\theta,\varphi) & \sqrt{\frac{1}{10}}Y_2^0(\theta,\varphi) & \sqrt{\frac{3}{5}}Y_2^{-2}(\theta,\varphi) \\ 0 & \sqrt{\frac{3}{10}}Y_2^1(\theta,\varphi) & \sqrt{\frac{3}{5}}Y_2^2(\theta,\varphi) & \sqrt{\frac{1}{10}}Y_2^0(\theta,\varphi) \end{pmatrix}$$
$$= \frac{\sqrt{2\pi}}{3}\mathcal{CT},$$

with the matrix $\mathcal{CT}$ (**c**oupling **t**erm), leads to the entire spin based "coupling" dynamic of this Hamiltonian.

As it currently stands, the Hamiltonian has the following form

$$\begin{aligned}\mathbf{H} =& \frac{1}{m}\left(\mathbf{p}_r^2 + \frac{\mathbf{L}^2}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) + \frac{1}{m^2}\left(V_r(r) + (\mathbf{p}_r^2 V_p(r))\right) \\ &+ \frac{1}{m^2}\left(2\mathbf{L}^2\left(V_L(r) + \frac{V_p(r)}{r^2}\right) - 2(\partial_r V_p(r))\partial_r + 2V_p(r)\mathbf{p}_r^2 + V_{\text{LS}}(r)\frac{\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2}{2} + V_{S_{12}}(r)\frac{\sqrt{2\pi}}{3}\mathcal{CT} + V_{S^2}(r)\frac{\mathcal{DT}}{4}\right).\end{aligned}$$

A closer examination of the above Hamiltonian will reveal that the states $|j, j_z; s; l\rangle$ are eigenstates of every termwith the sole exception of the $\mathcal{CT}$ term. This is the motivation for splitting the Hamiltonian $\mathbf{H}$ into a diagonal part

$$\begin{aligned}\mathbf{H}_{\text{D}} =& \frac{1}{m}\left(\mathbf{p}_r^2 + \frac{\mathbf{L}^2}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8\alpha^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) + \frac{1}{m^2}\left(V_r(r) + (\mathbf{p}_r^2 V_p(r))\right) \\ &+ \frac{1}{m^2}\left(2\mathbf{L}^2\left(V_L(r) + \frac{V_p(r)}{r^2}\right) - 2(\partial_r V_p(r))\partial_r + 2V_p(r)\mathbf{p}_r^2 + V_{\text{LS}}(r)\frac{\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2}{2} + V_{S^2}(r)\frac{\mathcal{DT}}{4}\right)\end{aligned}$$

and a non-diagonal component

$$\mathbf{H}_{\text{ND}} = \frac{V_{S_{12}}(r)}{m^2}\frac{\sqrt{2\pi}}{3}\mathcal{CT}.$$

At this point, it should be noted that a general state in the Hilbert space would exhibit an r dependent coefficient on which $\mathbf{p}_r$ and ∂_r will act.

2.3 Conservation Laws

To further simplify the problem, conservation laws are of great importance. Specifically, it will be shown that the following conservation laws hold.

Conservation Laws

The total angular momentum $\mathbf{J}^2$ as well as its z-component $\mathbf{J}_z$ and the total spin $\mathbf{S}^2$ are conserved, i.e. they commute with $\mathbf{H}$.
In particular, the state $\mathbf{H}|j, j_z; s; l\rangle$ decomposes into other $|j, j_z; s; l'\rangle$ with the same total angular momentum quantum numbers j and j_z as well as the same total spin quantum number s.
Furthermore, the parity and C parity remain unchanged.

It is now time to justify the above conservation laws. First of all, the operator $\mathbf{S}^2$ trivially commutes with $\mathbf{H}_{\text{D}}$ and it is straightforward to find the same for $\mathbf{H}_{\text{ND}}$. Moreover, it is also essentially trivial that $\mathbf{J}_z$ and $\mathbf{J}^2$ commute with $\mathbf{H}_{\text{D}}$ since $\mathbf{H}_{\text{D}}$ can be written as a linear combination of the commuting operators $\mathbf{J}^2, \mathbf{J}_z, \mathbf{S}^2$, and $\mathbf{S}_z$. It is a bit more tricky to prove that both $\mathbf{J}_z$ and $\mathbf{J}^2$ commute with $\mathbf{H}_{\text{ND}}$. This is done by recalling that

$$\mathbf{L}^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right), \qquad \mathbf{L}_x = \frac{1}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right),$$
$$\mathbf{L}_y = \frac{1}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad \mathbf{L}_z = \frac{1}{i}\frac{\partial}{\partial\varphi}.$$

(see for instance [4, (27.123) and (28.24)]) together with the explicit forms of $\mathbf{S}^2, \mathbf{S}_x, \mathbf{S}_y$, and $\mathbf{S}_z$ from the previous chapter compute the commutators $[\mathbf{J}_z, \mathbf{H}_{\text{ND}}]$ and $[\mathbf{J}^2, \mathbf{H}_{\text{ND}}]$.

Physically, it is clear that the Hamiltonian should preserve parity and C parity. To verify this mathematically, it suffices to consider the action of the non-diagonal component $\mathbf{H}_{\text{ND}}$ on the states $|j, j_z; s; l\rangle$. Observe that the

components of the state $|j, j_z; s; l\rangle \in \mathcal{H}^4$ consist solely of spherical harmonics with a total angular momentum of l. This implies that the components of $\mathcal{CT}|j, j_z; s; l\rangle$ are superpositions of the products

$$Y_l^m(\theta, \varphi) Y_2^n(\theta, \varphi)$$

with varying n and m. Using the contraction property of spherical harmonics, this product can be reexpressed as

$$Y_l^m(\theta, \varphi) Y_2^n(\theta, \varphi) = \sqrt{\frac{(2l+1)\cdot(2\cdot 2+1)}{4\pi}} \sum_{c=0}^{\infty} \sum_{\gamma=-c}^{c} (-1)^\gamma \sqrt{2c+1} \begin{pmatrix} l & 2 & c \\ m & n & -\gamma \end{pmatrix} \begin{pmatrix} l & 2 & c \\ 0 & 0 & 0 \end{pmatrix} Y_c^\gamma(\theta, \varphi),$$

where the terms resembling 2×3 matrices are Wigner 3j-symbols. A special property of the Wigner-3j-symbol is that

$$\begin{pmatrix} l & 2 & c \\ 0 & 0 & 0 \end{pmatrix}$$

is only non-zero if $l + 2 + c$ is even. This implies that c and l have the same parity (as integers), and thus the product $Y_l^m(\theta, \varphi) Y_2^n(\theta, \varphi)$ is a superposition of spherical harmonics with a total angular momentum of the same parity as l. This also shows that the total angular momenta of all the spherical harmonics appearing in the product $\mathcal{CT}|j, j_z; s; l\rangle$ have the same parity as l, and therefore the parity and C parity of this state are the same as those of $|j, j_z; s; l\rangle$ (compare the formula for parity and C parity).

2.4 Hamiltonian Acting on $|\psi\rangle$

After having split the Hamiltonian $\mathbf{H} = \mathbf{H}_\mathrm{D} + \mathbf{H}_\mathrm{ND}$, it is time to "solve" the time-independent Schrödinger equation $\mathbf{H}|\psi\rangle = E|\psi\rangle$ for a state $|\psi\rangle$ describing a particle of total momentum J, of parity P, and of charge parity C. In particular, the time-independent Schrödinger equation means that for all $j^{(1)}$, $j_z^{(1)}$, $s^{(1)}$, and $l^{(1)}$

$$\langle j^{(1)}, j_z^{(1)}; s^{(1)}; l^{(1)} E|\psi\rangle = \langle j^{(1)}, j_z^{(1)}; s^{(1)}; l^{(1)}|\mathbf{H}|\psi\rangle$$

$$\Leftrightarrow$$

$$\sum_{j_z^{(2)}=-J}^{J} \sum_{\substack{|s^{(2)}-l^{(2)}| \le J \le |s^{(2)}+l^{(2)}| \\ (-1)^{l^{(2)}+1}=P^{(2)} \\ (-1)^{l^{(2)}+s^{(2)}}=C^{(2)}}} A_{J, j_z^{(1)}; s^{(1)}; l^{(1)}}(r) \delta_{j^{(1)} J} \delta_{j_z^{(1)} j_z^{(2)}} \delta_{s^{(1)} s^{(2)}} \delta_{l^{(1)} l^{(2)}}$$

$$= \sum_{j_z^{(2)}=-J}^{J} \sum_{\substack{|s^{(2)}-l^{(2)}| \le J \le |s^{(2)}+l^{(2)}| \\ (-1)^{l^{(2)}+1}=P \\ (-1)^{l^{(2)}+s^{(2)}}=C}} \langle j^{(1)}, j_z^{(1)}; s^{(1)}; l^{(1)}|\mathbf{H}|J, j_z^{(2)}; s^{(2)}; l^{(2)}\rangle A_{J, j_z^{(2)}; s^{(2)}; l^{(2)}}(r),$$

where the representation of $|\psi\rangle$ in equation (2.1) was used. Due to the conservation laws from the previous section, the matrix element $\langle j^{(1)}, j_z^{(1)}; s^{(1)}; l^{(1)}|\mathbf{H}|j, j_z; s; l\rangle$ is zero unless $j^{(1)} = J$, $j_z^{(1)} = j_z^{(2)}$, and $s^{(1)} = s^{(2)}$. If the matrix element is zero, the above equality is trivially satisfied and nothing more needs to be done. Therefore, assume the equalities and let $j_z = j_z^{(1)} = j_z^{(2)}$, $s = s^{(1)} = s^{(2)}$, and $l = l^{(1)}$ (not necessarily $l^{(2)}$). The resulting equation is

$$A_{J, j_z; s; l}(r) \delta_{l l^{(2)}} = \sum_{\substack{|s-l^{(2)}| \le J \le |s+l^{(2)}| \\ (-1)^{l^{(2)}+1}=P \\ (-1)^{l^{(2)}+s}=C}} \langle J, j_z; s; l|\mathbf{H}|J, j_z; s; l^{(2)}\rangle A_{J, j_z; s^{(2)}; l^{(2)}}(r)$$

$$= \langle J, j_z; s; l|\mathbf{H}_\mathrm{D}|J, j_z; s; l\rangle A_{J, j_z; s; l}(r) + \sum_{\substack{|s-l^{(2)}| \le J \le |s+l^{(2)}| \\ (-1)^{l^{(2)}+1}=P \\ (-1)^{l^{(2)}+s}=C}} A_{J, j_z; s^{(2)}; l^{(2)}}(r) \langle J, j_z; s; l|\mathbf{H}_\mathrm{ND}|J, j_z; s; l^{(2)}\rangle.$$

Now, it is possible to substitute the full forms for the diagonal and non-diagonal part of Hamiltonian to obtain

$$\begin{aligned}EA_{J,0;s;l}(r) =&\left(\frac{1}{m}\left(-\partial_r^2-\frac{2}{r}\partial_r+\frac{l(l+1)}{r^2}\right)+\left(\frac{-e}{r}+\sigma r\right)+\frac{1}{m}\left(\frac{-8e^2}{9r^2}+\frac{2\sigma}{\pi}\ln(r)\right)+\frac{1}{m^2}\left(V_r(r)+\left(-\partial_r^2V_{\mathrm{p}}(r)-\frac{2}{r}\partial_rV_{\mathrm{p}}(r)\right)\right)\right.\\&+\frac{1}{m^2}\left(2l(l+1)\left(V_L(r)+\frac{V_{\mathrm{p}}(r)}{r^2}\right)-2(\partial_rV_{\mathrm{p}}(r))\partial_r+2V_{\mathrm{p}}(r)\left(-\partial_r^2-\frac{2}{r}\partial_r\right)+V_{\mathrm{LS}}(r)\frac{J(J+1)-l(l+1)-s(s+1)}{2}+V_{S^2}(r)\frac{\delta_{1s}-3\delta_{0s}}{4}\right)\Big)A_{J,0;s;l}(r)\\&+\sum_{\substack{|s-l'|\leq J\leq|s+l'|\\(-1)^{l'+1}=P\\(-1)^{l'+s}=C}}A_{J,0;s;l'}(r)\frac{V_{S_{12}}(r)}{m^2}\frac{\sqrt{2\pi}}{3}\langle J,0;s;l|\mathcal{CT}|J,0;s;l'\rangle\end{aligned}$$

for all s and l with $|l-s|\leq J\leq|l+s|$, $(-1)^{l+1}=P$ and $(-1)^{l+s}=C$. Here j_z was set to zero since the differential equation does not depend on it.

With a few considerations regarding the relationship between the parity and l as well as s, this implies the following.

Differential Equations General

Let J denote the total angular momentum, P the parity, and C the charge parity. Then four cases arise:

(i) If $J\geq 1$, $PC=1$, and $(-1)^J=P$, then

$$\begin{aligned}E\begin{pmatrix}A_{J,0;1;J-1}(r)\\A_{J,0;1;J+1}(r)\end{pmatrix}=&\left(\frac{1}{m}\left(-\partial_r^2-\frac{2}{r}\partial_r+\frac{1}{r^2}\begin{pmatrix}J\cdot(J-1)&0\\0&(J+1)(J+2)\end{pmatrix}\right)+\left(\frac{-e}{r}+\sigma r\right)+\frac{1}{m}\left(\frac{-8e^2}{9r^2}+\frac{2\sigma}{\pi}\ln(r)\right)+\frac{1}{m^2}\left(V_r(r)+\left(-\partial_r^2V_p(r)-\frac{2}{r}\partial_rV_p(r)\right)\right)\right.\\&+\frac{1}{m^2}\left(2\begin{pmatrix}J\cdot(J-1)&0\\0&(J+1)(J+2)\end{pmatrix}\left(V_L(r)+\frac{V_p(r)}{r^2}\right)-2(\partial_rV_p(r))\partial_r+2V_p(r)\left(-\partial_r^2-\frac{2}{r}\partial_r\right)+V_{\mathrm{LS}}(r)\begin{pmatrix}J-1&0\\0&-(J+1)\end{pmatrix}+\frac{V_{S^2}(r)}{4}\right)\\&\left.+\frac{V_{S_{12}}(r)}{m^2}\frac{\sqrt{2\pi}}{3}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle&\langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle\\\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle&\langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right)\begin{pmatrix}A_{J,0;1;J-1}(r)\\A_{J,0;1;J+1}(r)\end{pmatrix}.\end{aligned}$$

(ii) If $PC=1$ but $(-1)^J\neq P$, then

$$\begin{aligned}EA_{J,0;1;J}(r)=&\left(\frac{1}{m}\left(-\partial_r^2-\frac{2}{r}\partial_r+\frac{J(J+1)}{r^2}\right)+\left(\frac{-e}{r}+\sigma r\right)+\frac{1}{m}\left(\frac{-8e^2}{9r^2}+\frac{2\sigma}{\pi}\ln(r)\right)+\frac{1}{m^2}\left(V_r(r)+\left(-\partial_r^2V_p(r)-\frac{2}{r}\partial_rV_p(r)\right)\right)\right.\\&+\frac{1}{m^2}\left(2J(J+1)\left(V_L(r)+\frac{V_p(r)}{r^2}\right)-2(\partial_rV_p(r))\partial_r+2V_p(r)\left(-\partial_r^2-\frac{2}{r}\partial_r\right)+V_{\mathrm{LS}}(r)+\frac{V_{S^2}(r)}{4}\right)\Big)A_{J,0;1;J}(r)\\&+A_{J,0;1;J}(r)\frac{V_{S_{12}}(r)}{m^2}\frac{\sqrt{2\pi}}{3}\langle J,0;1;J|\mathcal{CT}|J,0;1;J\rangle.\end{aligned}$$

(iii) If $PC=-1$, then

$$\begin{aligned}EA_{J,0;0;J}(r)=&\left(\frac{1}{m}\left(-\partial_r^2-\frac{2}{r}\partial_r+\frac{J(J+1)}{r^2}\right)+\left(\frac{-e}{r}+\sigma r\right)+\frac{1}{m}\left(\frac{-8e^2}{9r^2}+\frac{2\sigma}{\pi}\ln(r)\right)+\frac{1}{m^2}\left(V_r(r)+\left(-\partial_r^2V_p(r)-\frac{2}{r}\partial_rV_p(r)\right)\right)\right.\\&+\frac{1}{m^2}\left(2J(J+1)\left(V_L(r)+\frac{V_p(r)}{r^2}\right)-2(\partial_rV_p(r))\partial_r+2V_p(r)\left(-\partial_r^2-\frac{2}{r}\partial_r\right)-V_{S^2}(r)\frac{3}{4}\right)\Big)A_{J,0;0;J}(r).\end{aligned}$$

(iv) If $J=0$, $P=1$, and $C=1$, then

$$\begin{aligned}EA_{0,0;1;1}(r)=&\left(\frac{1}{m}\left(-\partial_r^2-\frac{2}{r}\partial_r+\frac{2}{r^2}\right)+\left(\frac{-e}{r}+\sigma r\right)+\frac{1}{m}\left(\frac{-8e^2}{9r^2}+\frac{2\sigma}{\pi}\ln(r)\right)+\frac{1}{m^2}\left(V_r(r)+\left(-\partial_r^2V_p(r)-\frac{2}{r}\partial_rV_p(r)\right)\right)\right.\\&+\frac{1}{m^2}\left(4\left(V_L(r)+\frac{V_p(r)}{r^2}\right)-2(\partial_rV_p(r))\partial_r+2V_p(r)\left(-\partial_r^2-\frac{2}{r}\partial_r\right)-2V_{\mathrm{LS}}(r)+\frac{V_{S^2}(r)}{4}\right)\Big)A_{0,0;1;1}(r)\\&+A_{0,0;1;1}(r)\frac{V_{S_{12}}(r)}{m^2}\frac{\sqrt{2\pi}}{3}\langle 0,0;1;1|\mathcal{CT}|0,0;1;1\rangle.\end{aligned}$$

Here it was also used that $\langle J,0;0;J|\mathcal{CT}|J,0;0;J\rangle=0$.

The unknown still present in the established differential equation are the matrix elements $\langle J,0;s;l|\mathcal{CT}|J,0;s;l'\rangle$. These are computed with *Mathematica*, but it is important to be aware of the fact that the scalar product is only taken with respect to θ and φ which means that for the two general states

$$\begin{pmatrix}f_1(\theta,\varphi)\\f_2(\theta,\varphi)\\f_3(\theta,\varphi)\\f_4(\theta,\varphi)\end{pmatrix},\begin{pmatrix}g_1(\theta,\varphi)\\g_2(\theta,\varphi)\\g_3(\theta,\varphi)\\g_4(\theta,\varphi)\end{pmatrix}$$

it is defined as

$$\left\langle\begin{pmatrix}f_1(\theta,\varphi)\\f_2(\theta,\varphi)\\f_3(\theta,\varphi)\\f_4(\theta,\varphi)\end{pmatrix},\begin{pmatrix}g_1(\theta,\varphi)\\g_2(\theta,\varphi)\\g_3(\theta,\varphi)\\g_4(\theta,\varphi)\end{pmatrix}\right\rangle:=\int_{-\pi}^{\pi}d\,\varphi\int_0^{\pi}d\,\theta\sin(\theta)\sum_{i=1}^{4}f_i(\theta,\varphi)g_i^*(\theta,\varphi).$$

Keeping this in mind, the computation is straightforward and in Appendix B a table of the these matrix elements can be found.

Chapter 3

Asymptotic Behavior of the Differential Equation

The main aim of this thesis is to study the effects of different corrections to the Schrödinger equation governing bottomonium. More specifically, this thesis will study the following five cases of the Schrödinger equation within the static quark potential:

(i) Only the zeroth order in $1/m$.
(ii) Only the zeroth and first order in $1/m$.
(iii) The zeroth, the first, and the spin-dependent contributions.
(iv) The entire static quark potential but with $V_{\mathrm{p}}(r)$ set to 0.
(v) The complete static quark potential.

To solve any of these differential equations numerically, it is paramount to know the asymptotic behavior of the differential equation for $r \to 0$ and $r \to \infty$. Of these two situations, the case $r \to \infty$ is easily determined, namely due to the linear growth of the potential at ∞. This causes an exponential decay in all of the $A_{J,0;s;l}$, and thus $A_{J,0;s;l}$ must tend to zero as r tends to infinity.

The asymptotic behavior at $r = 0$ is, however, significantly more challenging to determine and will be the main focus of this chapter. The approach to solving for this asymptotic behavior is as follows. First, it is *assumed* that it is possible to neglect all but the dominating terms in the r dependent coefficients of $\partial_r^2 A_{J,0;s;l}(r)$, $\partial_r A_{J,0;s;l}(r)$, and $A_{J,0;s;l}(r)$. This means that if the coefficient is $1/r + 1/r^2$, only the $1/r^2$ term will be considered. It is important to emphasize that this is an assumption, not a necessity. This will result in a new differential equation which should ideally capture the asymptotic nature of the complete solution. In the non-coupled cases, this differential equation will be reducible to either the Bessel equation or the Cauchy-Euler differential equation. The coupled case requires slightly more dexterity. Here there is not an easy differential equation that comes to mind to which these coupled equations reduce to under substitution. Instead, the procedure will be very similar in how the Bessel and Cauch-Euler equations are solved. First, a substitution reminiscent of the one in the non-coupled case is carried out. Contrary to the non-coupled case this will not lead to an exact replica of, for example, the Bessel equation but will be close enough to utilize and generalize the techniques used in the solving of its non-coupled analogon to the coupled case. Once this is complete, the reduced differential equation, derived after neglecting non-dominating terms, is solved, but, importantly, it is *not* apparent that these solutions do in fact satisfy the assumption, i.e. all contributions of non-dominating terms of the coefficients of the derivatives in the limit $r \to 0$ are in fact negligible. It turns out that not all solutions of the reduced differential equation satisfy the assumption, and thus the number of asymptotic solutions is reduced to one for the non-coupled case and two for the coupled one, exactly as expected for a Schrödinger equation.

On a last note, in order to distinguish between the complete solution $A_{J,0;s;l}$ and the asymptotic solution for $r \to 0$ the latter will be decorated with $r \to 0$, i.e. $A^{r\to 0}_{J,0;s;l}(r)$.

3.1 Zeroth Order

In the zeroth order, the Schrödinger equation is simply

$$EA_{J,0;s;l}(r) = \left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right)\right)A_{J,0;s;l}(r).$$

To solve for the asymptotic behavior of the Schrödinger equation, it is necessary to distinguish between two cases, namely where $l = 0$, the so-called base case, and where $l \neq 0$.

Base Case ($l = 0$)

As described in the introduction of this chapter, assume that it is possible to neglect all in the limit r tends 0 non-dominating terms occurring in the coefficients of the derivatives of $A_{J,0;s;l}(r)$, leading to the differential equation

$$0 = \left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) - \frac{e}{r}\right) A^{r\to 0}_{J,0;s;0}(r).$$

After substituting

$$v = 2\sqrt{em}\sqrt{r} \qquad \text{and} \qquad A^{r\to 0}_{J,0;s;0}(r) = \frac{u(v)}{v},$$

rearranging results in to the modified Bessel equation

$$0 = v^2\partial_v^2 u(v) + v\partial_v u(v) + (v^2 - 1)u(v),$$

and $A^{r\to 0}_{J,0,s,0}(r)$ can be expressed as a linear superposition of

$$\sqrt{\frac{1}{r}}J_1(2\sqrt{em}\sqrt{r}) \qquad \text{and} \qquad \sqrt{\frac{1}{r}}Y_1(2\sqrt{em}\sqrt{r}), \tag{3.1}$$

where $J_1(x)$ and $Y_1(x)$ denote the Bessel function of order one as well as of first and second kind, respectively.

Under the assumption $A_{J,0;s,0}(r)$ should align with $A^{r\to 0}_{J,0;s,0}(r)$ in the limit, r tends to zero. It might be thought that this can be verified by simply plugging $A^{r\to 0}_{J,0;s,0}(r)$ into the original differential equation and then ensuring that under the limit r tending to 0 the two sides of the equation equate. Here a bit of caution needs to be exercised. Specifically, in its present form the original differential equation contains poles at zero in the coefficients of the derivatives. Replacing $A_{J,0;s,0}(r)$ with $A^{r\to 0}_{J,0;s,0}(r)$ resembles the swapping of limits with $1/r$. In general this is not possible. To alleviate this problem, it is useful to first multiply the entire original Schrödinger differential equation by a power of r such that no poles are present in the coefficients of the derivatives (poles can be present in the coefficient of plain $A_{J,0;s,0}(r)$). In this case, it suffices to muliply with r, leading to the equation

$$\begin{aligned} 0 &= \left(\frac{r}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) + -e + \sigma r^2 - Er\right) A^{r\to 0}_{J,0,s,0}(r) \\ &= \left(\sigma r^2 - Er\right) A^{r\to 0}_{J,0,s,0}(r) \qquad \text{for } r \to 0. \end{aligned}$$

Here it is advantageous to multiply with the smallest power of r which alleviates poles because this gives the best chance of eliminating asymptotic solutions $A^{r\to 0}_{J,0,s,0}(r)$ which cannot lead to a solution of the full differential equation.

Returning to the specific situation at hand, it is well-known that the Bessel function of the second kind $Y_1(x)$ diverges to $-\infty$ for x tending to infinity and is thus unequal to zero. This remains unchanged when adding the decorations present in equation (3.1) and even when multiplying with $\sigma r^2 - Er$. Therefore,

$$\sqrt{\frac{1}{r}}Y_1(2\sqrt{em}\sqrt{r})$$

cannot be an asymptotic solution. Similarly, it can be shown that

$$\sqrt{\frac{1}{r}}J_1(2\sqrt{em}\sqrt{r})$$

has no such grievances and is thus the rightful aysmptotic solution, i.e.

$$A^{r\to 0}_{J,0,s,0}(r) = B \cdot \sqrt{\frac{1}{r}}J_1(2\sqrt{em}\sqrt{r}) \qquad \text{for } B \in \mathbb{C}.$$

General Case $(l \neq 0)$

Again assume that the non-dominating terms in the coefficients of $\partial_r^2 A_{J,0;s;l}$, $\partial_r A_{J,0;s;l}$, and $A_{J,0;s;l}$ can safely be neglected, leading to

$$0 = \frac{1}{m}\bigg(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\bigg) A^{r\to 0}_{J,0;s;l}(r)$$

or

$$0 = r^2\partial_r^2 A^{r\to 0}_{J,0;s;l}(r) + 2r\partial_r A^{r\to 0}_{J,0;s;l}(r) - l(l+1)A^{r\to 0}_{J,0;s;l}(r).$$

This is a Cauchy-Euler differential equation whose solution are well-known to be linear combinations of $r^{\lambda_{1,2}}$ where $\lambda_\pm$ are the zeros of

$$x^2 + x - l(l+1).$$

This means that $\lambda_+ = l$ and $\lambda_- = -l-1$. As before, the assumption, in this case

$$0 = (-e + \sigma r^2 - Er)A^{r\to 0}_{J,0;s;l}(r),$$

needs to be verified, and it is immediately clear that only

$$A^{r\to 0}_{J,0;s;l}(r) = Cr^l \qquad \text{for } C \in \mathbb{C}$$

is a valid asymptotic solution.

3.2 First Order

The situation for the first order is very similar to that of the zeroth order. Now, the full Schrödinger equation is

$$EA_{J,0;s;l}(r) = \bigg(\frac{1}{m}\bigg(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\bigg) + \bigg(\frac{-e}{r} + \sigma r\bigg) + \frac{1}{m}\bigg(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\bigg)\bigg)A_{J,0;s;l}(r).$$

Assuming that it is possible to neglect all terms, which are not dominant as r tends to 0, results in another Cauchy-Euler equation

$$0 = \frac{1}{m}\bigg(-\partial_r^2 - \frac{2}{r}\partial_r + \bigg(l(l+1) + \frac{-8e^2}{9}\bigg)\frac{1}{r^2}\bigg) A^{r\to 0}_{J,0;s;l}(r)$$

or

$$0 = r^2\partial_r^2 A^{r\to 0}_{J,0;s;l}(r) + 2r\partial_r A^{r\to 0}_{J,0;s;l}(r) - \bigg(l(l+1) + \frac{-8e^2}{9}\bigg) A^{r\to 0}_{J,0;s;l}(r).$$

Here it was used that $1/r^2$ dominates $\ln(r)$ for r tending to 0. The solutions are now

$$A^{r\to 0}_{J,0;s;l}(r) = r^{\lambda_\pm}$$

with $\lambda_\pm$ being the zeros of the polynomial $x^2 + x - l(l+1) + 8e^2/9$, i.e.

$$\lambda_\pm = \frac{-1}{2} \pm \frac{1}{2}\sqrt{1 + 4\bigg(l(l+1) - \frac{8e^2}{9}\bigg)}.$$

This time the postulation works out to be

$$0 = \bigg(-e + \sigma r^2 + \frac{2\sigma}{\pi m} r\ln(r) - Er\bigg) A^{r\to 0}_{J,0;s;l}(r) \qquad \text{for } r \to 0,$$

and since $\lambda_- < 0$ this can only be satisfied by

$$A^{r\to 0}_{J,0;s;l} = B \cdot x^\lambda \qquad \text{with } \lambda = \frac{-1}{2} + \frac{1}{2}\sqrt{1 + 4\bigg(l(l+1) - \frac{8e^2}{9}\bigg)} \text{ and } B \in \mathbb{C}.$$

3.3 Spin-dependent Contribution

The moment the full static quark potential is considered more fundamentally different, cases arise which need to be individually studied. Specifically, the following cases are relevant:

(i) Base Case ($J^{PC} = 0^{-+}$) and $S = 0$
(ii) Non-Coupled Case
(iii) Coupled Case

The asymptotic behavior in each of these cases will now be discussed individually in the following subsections. For all the cases, only the spin-dependent contribution to the static quark potential will be considered. This is the same as simply presuming the following forms for the radial potentials:

Explicit Forms for Radial Potentials

$$V_{\mathrm{LS}}(r) = \frac{5e}{2r^3} + \frac{\sigma}{2r} \qquad V_{\mathrm{r}}(r) = 0$$
$$V_{S_{12}}(r) = \frac{3e}{r^3} \qquad V_{\mathrm{L}}(r) = 0$$
$$V_{S^2}(r) = 0 \qquad V_{\mathrm{p}}(r) = 0$$

Base Case ($J^{PC} = 0^{-+}$) and $S = 0$

If $s = 0$, then all terms of order $1/m^2$ disappear, and the Schrödinger equation is

$$0 = \left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) - E\right) A_{J,0;0;l}(r).$$

This is however identical to the first order case in the previous section, and therefore the asymptotic behavior is yet again

$$A^{r\to 0}_{J,0;0;l} = B \cdot x^{\lambda} \qquad \text{with } \lambda = \frac{-1}{2} + \frac{1}{2}\sqrt{1 + 4\left(l(l+1) - \frac{8e^2}{9}\right)} \text{ and } B \in \mathbb{C}.$$

Non-Coupled Case excluding $J^{PC} = 0^{-+}$ and $S = 0$

For the general non-coupled case, the differential equations are once again no longer Cauchy-Euler differential equations. They are

$$\begin{aligned} 0 = &\left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right)\right. \\ &\left. + \frac{1}{m^2}\left(\left(\frac{5e}{2r^3} + \frac{\sigma}{2r}\right)\frac{J(J+1) - l(l+1) - s(s+1)}{2} + \frac{3e}{r^3}\frac{\sqrt{2\pi}}{3}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right) - E\right) A_{J,0;s;l}(r). \end{aligned}$$

As before, assume that it is permissible to neglect all but the dominant terms in the coefficients of $\partial_r^2 A_{J,0;s;l}(r)$, $\partial_r A_{J,0;s;l}(r)$, and $A_{J,0;s;l}(r)$, resulting in

$$0 = \frac{-1}{m}\partial_r^2 A^{r\to 0}_{J,0;s;l}(r) - \frac{2}{rm}\partial_r A^{r\to 0}_{J,0;s;l}(r) + \frac{1}{m^2}\left(\frac{5e(J(J+1) - l(l+1) - s(s+1))}{4} + e\sqrt{2\pi}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right)\frac{1}{r^3}A^{r\to 0}_{J,0;s;l}(r)$$

or

$$0 = \partial_r^2 A^{r\to 0}_{J,0;s;l}(r) + \frac{2}{r}\partial_r A^{r\to 0}_{J,0;s;l}(r) - \frac{1}{m}\left(\frac{5e(J(J+1) - l(l+1) - s(s+1))}{4} + e\sqrt{2\pi}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right)\frac{1}{r^3}A^{r\to 0}_{J,0;s;l}(r).$$

From now on, the constant appearing in front of $A_{J,0;s;l}(r)/r^3$ will be abbreviated with C^{SDC}_{NCC} (**s**pin-**d**ependent **c**ase, **n**on-**c**ouping **c**ase). It will now be assumed that C^{SDC}_{NCC} is unequal to zero which will later be verified numerically.

If C^{SDC}_{NCC} is positive, then making the substitution

$$v = 2\sqrt{C^{SDC}_{NCC}}\sqrt{\frac{1}{r}} \qquad \text{and} \qquad A^{r\to 0}_{J,0;s;l}(r) = \frac{1}{2}vu(v)$$

with subsequent rearranging leads to the Bessel equation

$$0 = v^2\partial_v^2 u(v) + v\partial_v u(v) + \left(v^2 - 1^2\right)u(v),$$

and thus $A^{r\to 0}_{J,0;s;l}(r)$ has the two linear independent solutions

$$\sqrt{C^{SDC}_{NCC}}\sqrt{\frac{1}{r}}J_1\left(2\sqrt{C^{SDC}_{NCC}}\sqrt{\frac{1}{r}}\right) \quad \text{and} \quad \sqrt{C^{SDC}_{NCC}}\sqrt{\frac{1}{r}}Y_1\left(2\sqrt{C^{SDC}_{NCC}}\sqrt{\frac{1}{r}}\right).$$

Similarly, if C^{SDC}_{NCC} is negative, then the substitution

$$v = 2\sqrt{-C^{SDC}_{NCC}}\sqrt{\frac{1}{r}} \qquad \text{and} \qquad A_{J,0;s;l}(r) = \frac{1}{2}vu(v)$$

does the trick, arriving at to the modified Bessel equation

$$0 = v^2\partial_v^2 u(v) + v\partial_v u(v) - \left(v^2 + 1^2\right) u(v)$$

with the two linear independent solutions

$$\sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}I_1\left(2\sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}\right) \quad \text{and} \quad \sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}K_1\left(2\sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}\right)$$

of $A^{r\to 0}_{J,0;s;l}(r)$ being composed of the modified Bessel functions of order one and of the first and second type.

Now, it is time to consider the assumption

$$0 = \left(\frac{l(l+1)}{m}\frac{1}{r} + \left(-e + \sigma r^2\right) + \frac{1}{m}\left(\frac{-8e^2}{9r} + \frac{2\sigma}{\pi} r\ln(r)\right) + \frac{1}{m^2}\frac{\sigma}{2}\frac{J(J+1) - l(l+1) - s(s+1)}{2} - Er\right) A_{J,0;s;l}(r)$$

for $r \to 0$.

If C_{NCC}^{SDC} is negative, then only the solution

$$\sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}K_1\left(2\sqrt{-C_{NCC}^{SDC}}\sqrt{\frac{1}{r}}\right)$$

comes into question. If C_{NCC}^{SDC} is however positive, then the limit $r \to 0$ is not defined for either of the two solutions. Thus, neither can be excluded. Interestingly, this is not going to be a problem for the numerical computation of the energy state as both will lead to the same result. Why this is the case is an open question.

Coupled Case

Solving the coupled case is slightly more elaborate since there is no obvious closed form. First, the differential equation is

$$\begin{aligned}0 =& \left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix}\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right)\right. \\ &\left. + \frac{1}{m^2}\left(\left(\frac{5e}{2r^3} + \frac{\sigma}{2r}\right)\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix} + \frac{3e}{r^3}\frac{\sqrt{2\pi}}{3}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right) - E\right)\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}\end{aligned}$$

which under the assumption simplifies to

$$\begin{aligned}0 =& \frac{-1}{m}\begin{pmatrix}\partial_r^2 A^{r\to 0}_{J,0;1;J-1}(r) \\ \partial_r^2 A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix} - \frac{2}{rm}\begin{pmatrix}\partial_r A^{r\to 0}_{J,0;1;J-1}(r) \\ \partial_r A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix} \\ &+ \frac{1}{m^2}\left(\frac{5e}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix} + e\sqrt{2\pi}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right)\frac{1}{r^3}\begin{pmatrix} A^{r\to 0}_{J,0;1;J-1}(r) \\ A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix}\end{aligned}$$

or

$$\begin{aligned}0 =& \begin{pmatrix}\partial_r^2 A^{r\to 0}_{J,0;1;J-1}(r) \\ \partial_r^2 A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix} + \frac{2}{r}\begin{pmatrix}\partial_r A^{r\to 0}_{J,0;1;J-1}(r) \\ \partial_r A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix} \\ &- \frac{1}{m}\left(\frac{5e}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix} + e\sqrt{2\pi}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right)\frac{1}{r^3}\begin{pmatrix} A^{r\to 0}_{J,0;1;J-1}(r) \\ A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix}.\end{aligned}$$

From now on, the matrix in front of $1/r^3$ will be denoted with C_{CC}^{SDC} (**s**pin-**d**ependent **c**ase, **c**ouping **c**ase). Additionally, the matrix C_{CC}^{SDC} will henceforth be assumed to be invertible, a fact that will be verified numerically later on.

Taking inspiration from the non-coupled case, the substitution

$$v = \sqrt{\frac{1}{r}} \qquad \text{and} \qquad \begin{pmatrix} A^{r\to 0}_{J,0;1;J-1}(r) \\ A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix} = v\begin{pmatrix} u_-(v) \\ u_+(v)\end{pmatrix} = vu(v)$$

is now carried out, resulting in

$$0 = v^2\partial_v^2 u + v\partial_v u - (1 - 4C_{CC}^{SDC}v^2)u(v).$$

This differential equation now lends itself to the Frobenius method, i.e. making the ansatz that the linear independent solutions of the above equations can be expressed as a power series in v with a multiplicative factor v^λ as

$$v^\lambda\sum_{k=0}^{\infty}\begin{pmatrix} a_k \\ b_k\end{pmatrix}v^k = v^\lambda\sum_{k=0}^{\infty} c_k v^k,$$

where $c_0 \neq 0$. Substituting this ansatz into the differential equation leads to

$$0 = \sum_{k=0}^{\infty}((k+\lambda)(k+\lambda-1)c_k v^{k+\lambda} + \sum_{k=0}^{\infty}(k+\lambda)c_k v^{k+\lambda} - \sum_{k=0}^{\infty} c_k v^{k+\lambda} + \sum_{k=2}^{\infty} 4C_{CC}^{SDC} c_{k-2} v^{k+\lambda}.$$

In particular, the indicial polynomial is $(\lambda(\lambda-1)+\lambda-1)$, yielding to $\lambda = \pm 1$ which implies that $c_1 = 0$. Furthermore, the recursive relationship is

$$((k+\lambda)^2 - 1)c_k = -4C_{CC}^{SDC} c_{k-2} \text{ for } k \geq 2.$$

For $\lambda = 1$, this comes to a power series where every coefficient depends solely on c_0. However, for $\lambda = -1$ the above equality implies for $k = 2$ that

$$((2-1)^2 - 1)c_2 = -4C_{CC}^{SDC} c_0$$

or $c_0 = 0$ since C_{CC}^{SDC} is assumed to be invertible. This contradicts the above assumption that $c_0 \neq 0$ and, therefore, no new solutions arise.

Up until this point, only two linear independent solutions to the above differential equation have been found, namely those resulting from the above recursive relationship with $\lambda = 1$ as well as $c_0 = (1, 0)$ or $c_0 = (0, 1)$. Conventional wisdom would dictate that four linear independent solutions exist. Taking inspiration from the Fuchs theorem regarding the equivalent one-dimensional problem make the ansatz

$$\begin{aligned} u(v) &= \ln(v) \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} v^{k+1} + \sum_{k=0}^{\infty} \begin{pmatrix} d_k \\ e_k \end{pmatrix} v^{k+\mu} \\ &= \ln(v) q(v) + \sum_{k=0}^{\infty} f_k v^{k+\mu}, \end{aligned}$$

where $f_0 \neq 0$, and consequently

$$q(v) = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} v^{k+1} = v^{\lambda} \sum_{k=0}^{\infty} c_k v^{k+1}$$

is one of the already solution found solutions. First, notice that the derivatives of $\ln(v)q(v)$ are

$$\partial_v(\ln(v)q(v)) = \frac{q(v)}{v} + \ln(v)\partial_v q(v)$$

and

$$\partial_v^2(\ln(v)q(v)) = \frac{-q(v)}{v^2} + \frac{2\partial_v q(v)}{v} + \ln(v)\partial_v^2 q(v).$$

Substituting the above ansatz into the differential equation yields

$$\begin{aligned} 0 =& v^2 \left(\frac{-q(v)}{v^2} + \frac{2\partial_v q(v)}{v} + \ln(v)\partial_v^2 q(v) + \sum_{k=0}^{\infty}(k+\mu)(k+\mu-2) f_k v^{k+\mu-2} \right) u \\ &+ v \left(\frac{q(v)}{v} + \ln(v)\partial_v q(v) + \sum_{k=0}^{\infty}(k+\mu) f_k v^{k+\mu-1} \right) \\ &- (1 - 4C_{CC}^{SDC} v^2) \left(\ln(v) q(v) + \sum_{k=0}^{\infty} f_k v^{k+\mu} \right) \\ =& 2v\partial_v q(v) + \ln(v)(v^2\partial_v^2 q(v) + v\partial_v q(v) - (1 - 4C_{CC}^{SDC} v^2) q(v)) \\ &+ \sum_{k=0}^{\infty}(k+\mu)(k+\mu-1) f_k v^{k+\mu} + \sum_{k=0}^{\infty}(k+\mu) f_k v^{k+\mu} - \sum_{k=0}^{\infty} f_k v^{k+\mu} + \sum_{k=2}^{\infty} 4C_{CC}^{SDC} f_{k-2} v^{k+\mu} \\ =& \sum_{k=0}^{\infty} 2(k+1) c_k v^{k+1} + \sum_{k=0}^{\infty}(k+\mu)(k+\mu-1) f_k v^{k+\mu} + \sum_{k=0}^{\infty}(k+\mu) f_k v^{k+\mu} - \sum_{k=0}^{\infty} f_k v^{k+\mu} \\ &+ \sum_{k=2}^{\infty} 4C_{CC}^{SDC} f_{k-2} v^{k+\mu}. \end{aligned}$$

Here it was made use of that $q(v)$ already satisfies the differential equation. Observe that the coefficient $2c_0$ of v^2 in the first power series is per assumption unequal to zero. This implies that it must cancel out another coefficient of v^2 in one of the other power series. In particular, μ must be an integer and no greater than 1. First,

assume that $\mu = 1$. Comparing coefficients for $k = 0$ then results in

$$0 = 2c_0 + (1^2 - 1)f_0 = 2c_0,$$

which is a contradiction since $c_0 \neq 0$. If $\mu < 1$, then the lowest order coefficient is simply

$$(\mu^2 - 1)f_0,$$

and μ must be -1. Comparing coefficients again leads to the solution

$$u(v) = \ln(v)\sum_{k=0}^{\infty} c_k v^{k+1} + \sum_{k=0}^{\infty} f_k v^{k-1}$$

with the recursive relationships

$$\begin{aligned} c_1 &= 0, & f_0 &= -\frac{(C_{CC}^{SDC})^{-1}}{2}c_0, \\ & & f_1 &= 0, \\ c_k &= \frac{-4C_{CC}^{SDC}}{(k+1)^2-1}c_{k-2} \qquad \text{for } k \geq 2, & f_k &= \frac{-4C_{CC}^{SDC}}{(k-1)^2-1}f_{k-2} - \frac{2((k-2)+1)}{(k-1)^2-1}c_{k-2} \\ & & & \text{for } k \geq 3. \end{aligned}$$

Notice that in the above collection of recursive relations both c_0 and f_2 are undetermined. Since all the relations are linear, this results in *four* linear independent solutions with

(i) $c_0 = (1,0)$ and $f_2 = (0,0)$,
(ii) $c_0 = (0,1)$ and $f_2 = (0,0)$,
(iii) $c_0 = (0,0)$ and $f_2 = (1,0)$,
(iv) $c_0 = (0,0)$ and $f_2 = (0,1)$.

This may seem like a contradiction to the conventional wisdom that the two-dimensional second order differential equation should in total have four solutions and not four plus two (as originally found). However, this is easily resolved by noting that $c_0 = 0$ as well as $f_2 = (1,0)$ and $f_2 = (0,1)$ lead to the original solutions found above.

To summarize, the four linear independent solutions for

$$\begin{pmatrix} A_{J,0;1;J-1}^{r\to 0}(r) \\ A_{J,0;1;J+1}^{r\to 0}(r) \end{pmatrix}$$

are given by

$$\frac{-1}{2}\sqrt{\frac{1}{r}}\ln(r)\sum_{k=0}^{\infty} c_k r^{-(k+1)/2} + \sqrt{\frac{1}{r}}\sum_{k=0}^{\infty} f_k r^{-(k-1)/2}$$

with the recursive relationships

$$\begin{aligned} c_1 &= 0, & f_0 &= -\frac{(C_{CC}^{SDC})^{-1}}{2}c_0, \\ & & f_1 &= 0, \\ c_k &= \frac{-4C_{CC}^{SDC}}{(k+1)^2-1}c_{k-2} \qquad \text{for } k \geq 2, & f_k &= \frac{-4C_{CC}^{SDC}}{(k-1)^2-1}f_{k-2} - \frac{2((k-2)+1)}{(k-1)^2-1}c_{k-2} \\ & & & \text{for } k \geq 3 \end{aligned}$$

when

(i) $c_0 = (1,0)$ and $f_2 = (0,0)$,
(ii) $c_0 = (0,1)$ and $f_2 = (0,0)$,
(iii) $c_0 = (0,0)$ and $f_2 = (1,0)$,
(iv) $c_0 = (0,0)$ and $f_2 = (0,1)$.

All that remains is to verify the assumption which is

$$0 = \left(\frac{1}{m}\frac{1}{r}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2) \end{pmatrix} + (-e+\sigma r^2) + \frac{1}{m}\left(\frac{-8e^2}{9r} + \frac{2\sigma}{\pi}\ln(r)r\right) + \frac{1}{m^2}\left(\frac{\sigma}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1) \end{pmatrix}\right) - Er\right)\begin{pmatrix} A_{J,0;1;J-1}^{r\to 0}(r) \\ A_{J,0;1;J+1}^{r\to 0}(r) \end{pmatrix}$$

for $r \to 0$.

This condition can now be used to numerically find which of the four presented solutions are in fact the asymptotic ones.

3.4 Complete Static Quark Potential with $V_{\mathbf{p}}(r) = 0$

The complete static quark potential with the exception that $V_{\mathrm{p}}(r)$ is set to zero only differs from the case using a exclusively spin-dependent contribution in the additional term $\frac{1}{m^2}2l(l+1)V_{\mathrm{L}}(r)$. Specifically, the following potentials come into play in this case:

Explicit Forms for Potentials (due to [10], [1], [6])

$$\begin{aligned} V_{\mathrm{LS}}(r) &= \frac{5e}{2r^3} + \frac{\sigma}{2r} & V_{\mathrm{r}}(r) &= 0 \\ V_{S_{12}}(r) &= \frac{3e}{r^3} & V_{\mathrm{L}}(r) &= \frac{e}{4r^3} - \frac{\sigma}{12r} \\ V_{S^2}(r) &= 0 & V_{\mathrm{p}}(r) &= 0 \end{aligned}$$

This changes a few constants in comparison with the above case, but the mathematical calculations are essentially identical. So this section will mainly just present the differences to the asymptotic solutions in the previous section.

Base Case ($J^{PC} = 0^{-+}$)

Since $l = 0$, the additional term is zero, and the base case is again identical to the first order case, i.e.

$$A_{0,0;0;0} = Cx^{\lambda} \qquad \text{with} \qquad \lambda = \frac{1}{2} + \frac{-1}{2}\sqrt{1 - 4\frac{8e^2}{9}} \text{ and } C \in \mathbb{C}.$$

Notice that the case where $s = 0$ is no longer included here since the additional terms ensure that the $1/m^2$ term does not disappear.

Non-Coupled Case excluding $J^{PC} = 0^{-+}$

In comparison to the non-coupled case for exclusively spin-dependent contributions, the present case only possesses the additional factor

$$\frac{1}{m^2}2l(l+1)V_{\mathrm{L}}(r) = \frac{1}{m^2}l(l+1)\left(\frac{e}{2r^3} - \frac{\sigma}{6r}\right)$$

or with regard to the assumption simply

$$\frac{1}{m^2}l(l+1)\frac{e}{2r^3}.$$

Adding this factor to the non-coupled case from the previous section amounts to considering the constant

$$C_{NCC}^{SSQ} = -\frac{1}{m}\left(\frac{el(l+1)}{2} + \frac{5e(J(J+1) - l(l+1) - s(s+1))}{4} + e\sqrt{2\pi}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right)$$

(**s**implified **s**tatic **q**uark potential, **n**on-**c**ouping **c**ase).

In particular, this means that if C_{NCC}^{SSQ} is positive, then again both solutions

$$C\sqrt{\frac{1}{r}}J_1\left(2\sqrt{C_{NCC}^{SSQ}}\sqrt{\frac{1}{r}}\right) \qquad \text{and} \qquad C\sqrt{\frac{1}{r}}Y_1\left(2\sqrt{C_{NCC}^{SSQ}}\sqrt{\frac{1}{r}}\right) \qquad \text{for } C \in \mathbb{C}$$

come into question, and for a negative C_{NCC}^{SSQ} the asymptotic solution is

$$C\sqrt{\frac{1}{r}}K_1\left(2\sqrt{-C_{NCC}^{SSQ}}\sqrt{\frac{1}{r}}\right) \qquad \text{for } C \in \mathbb{C}.$$

Coupled Case

The coupled case works very similar to the non-coupled case in comparison with the previous section, namely the only modification necessary to the previous solution is to alter the matrix C_{CC}^{SDC} to contain the new term $2l(l+1)V_L(r)/m^2$. The new matrix then works out to be

$$C_{CC}^{SSQ} = -\frac{1}{m}\left(\frac{e}{2}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2) \end{pmatrix} + \frac{5e}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1) \end{pmatrix} + e\sqrt{2\pi}\begin{pmatrix} \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle \end{pmatrix}\right)$$

(**s**implified **s**tatic **q**uark potential, **c**oupled **c**ase).

The solution to the coupled case is now derived analogously to the previous coupled case, and as such four linear independent solutions

$$\begin{pmatrix} A^{r\to 0}_{J,0;1;J-1}(r) \\ A^{r\to 0}_{J,0;1;J-1}(r) \end{pmatrix} = \frac{-1}{2}\sqrt{\frac{1}{r}}\ln(r)\sum_{k=0}^{\infty} c_k r^{-(k+1)/2} + \sqrt{\frac{1}{r}}\sum_{k=0}^{\infty} f_k r^{-(k-1)/2}$$

arise from the recursive relationship

$$\begin{aligned} c_1 &= 0, & f_0 &= -\frac{(C^{SDC}_{CC})^{-1}}{2}c_0, \\ & & f_1 &= 0, \\ c_k &= \frac{-4C^{SDC}_{CC}}{(k+1)^2-1}c_{k-2} \quad \text{for } k \geq 2, & f_k &= \frac{-4C^{SDC}_{CC}}{(k-1)^2-1}f_{k-2} - \frac{2((k-2)+1)}{(k-1)^2-1}c_{k-2} \\ & & & \text{for } k \geq 3 \end{aligned}$$

with each corresponding to one of the following starting data

(i) $c_0 = (1,0)$ and $f_2 = (0,0)$,
(ii) $c_0 = (0,1)$ and $f_2 = (0,0)$,
(iii) $c_0 = (0,0)$ and $f_2 = (1,0)$,
(iv) $c_0 = (0,0)$ and $f_2 = (0,1)$.

The additional term in comparison with the purely spin-dependent case in the previous section further changes the asymptotic condition to

$$0 = \left(\frac{1}{m}\frac{1}{r}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix} + (-e+\sigma r^2) + \frac{1}{m}\left(\frac{-8e^2}{9r} + \frac{2\sigma}{\pi}\ln(r)r\right) + \frac{1}{m^2}\left(\frac{-\sigma r}{6}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix} + \frac{\sigma}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix}\right) - Er\right)\begin{pmatrix} A^{r\to 0}_{J,0;1;J-1}(r) \\ A^{r\to 0}_{J,0;1;J+1}(r)\end{pmatrix}$$

for $r \to 0$.

As before, this will be applied to numerically find the relevant asymptotic solutions.

3.5 Complete Static Quark Potential

The complete static quark potential is substantially different from the two previous cases since it contains two additional derivatives obtained from a non-zero $V_{\mathrm{p}}(r)$. Specifically, this section will use the following potentials.

Explicit Forms for Potentials (due to [10], [1], [6])

$$\begin{aligned} V_{\mathrm{LS}}(r) &= \frac{5e}{2r^3} + \frac{\sigma}{2r} & V_{\mathrm{r}}(r) &= 0 \\ V_{S_{12}}(r) &= \frac{3e}{r^3} & V_{\mathrm{L}}(r) &= \frac{e}{4r^3} - \frac{\sigma}{12r} \\ V_{S^2}(r) &= 0 & V_{\mathrm{p}}(r) &= -\frac{C_F\alpha_s\mu}{2\pi} - \frac{e}{2r} \end{aligned}$$

Base Case ($J^{PC} = 0^{-+}$)

In the base case, the Schrödinger equation is

$$\begin{aligned} 0 = &\left(\frac{1}{m}\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) + \frac{1}{m^2}\left(-\frac{-e}{r^3} - \frac{2}{r}\frac{e}{2r^2}\right)\right. \\ &\left. + \frac{1}{m^2}\left(-2\left(\frac{e}{2r^2}\right)\partial_r + 2\left(-\frac{C_F\alpha_s\mu}{2\pi} - \frac{e}{2r}\right)\left(-\partial_r^2 - \frac{2}{r}\partial_r\right)\right) - E\right)A_{0,0;0;0}(r) \end{aligned}$$

and after grouping like terms

$$0 = \left(\frac{-1}{m} + \frac{C_F\alpha_s\mu}{\pi m^2} + \frac{e}{rm^2}\right)\partial_r^2 A_{J,0;s;l}(r) + \left(\frac{-2}{mr} + \frac{-e}{r^2m^2} + \frac{C_F\alpha_s\mu}{2\pi m^2}\frac{4}{r} + \frac{2e}{r^2m^2}\right)\partial_r A_{J,0;s;l}(r) + \left(\left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right) - E\right)A_{0,0;0;0}(r).$$

Next, assume that all but the dominant terms around $r = 0$ can be ignored. This leaves just

$$0 = \frac{e}{rm^2}\partial_r^2 A^{r\to 0}_{J,0;s;l}(r) + \frac{e}{r^2m^2}\partial_r A^{r\to 0}_{J,0;s;l}(r) + \frac{1}{m}\frac{-8e^2}{9r^2}A^{r\to 0}_{0,0;0;0}(r)$$

or

$$0 = r\partial_r^2 A^{r\to 0}_{0,0;0;0}(r) + \partial_r A^{r\to 0}_{J,0;s;l}(r) + \frac{-8me^2}{9e}A^{r\to 0}_{0,0;0;0}(r).$$

For notational unity, let $C_{SQP}^{BC} = \frac{-8me^2}{9e}$ (**s**tatic **q**uark **p**otential, **b**ase **c**ase). In order to solve this differential equation, it is useful to first substitute

$$v = 2\sqrt{-C_{SQP}^{BC} r} \qquad A_{0,0;0;0}^{r\to 0}(r) = \frac{u(v)}{v^2}.$$

Under this transformation, the differential equation becomes

$$0 = v^2 \partial_v^2 u + v\partial_v u - v^2 u(v),$$

which is the modified Bessel equation. Thus, $A_{0,0;0;0}^{r\to 0}(r)$ can be written as a linear superposition of

$$I_0\left(2\sqrt{-C_{SQP}^{BC} r}\right) \qquad \text{and} \qquad K_0\left(2\sqrt{-C_{SQP}^{BC} r}\right).$$

Once again, the assumption, in this case

$$0 = \left(\frac{-r^2}{m} + \frac{C_F\alpha_s\mu r^2}{\pi m^2}\right)\partial_r^2 A_{J,0;s;l}^{r\to 0}(r) + \left(\frac{-2r}{m} + 4\frac{C_F\alpha_s\mu r}{2\pi m^2}\right)\partial_r A_{J,0;s;l}^{r\to 0}(r) + \left(\left(-er + \sigma r^3\right) + \frac{1}{m}\left(\frac{2\sigma}{\pi} r^2 \ln(r)\right) - Er^2\right) A_{0,0;0;0}^{r\to 0}(r)$$

for $r \to 0$,

will eliminate one of these solutions, namely the one containing the modified Bessel function of the second kind K_0. This is because the derivative of $K_0(2\sqrt{-C_{SQP}^{BC} r})$ times r as $r \to 0$ is unequal to zero, while the derivatives of $I_0(2\sqrt{-C_{SQP}^{BC} r})$ converge to zero as r tends to 0, even after scaling with r and r^2, respectively. Therefore, the final solution is

$$A_{0,0;0;0}(r) = CI_0\left(2\sqrt{-C_{SQP}^{BC} r}\right) \qquad \text{for } C \in \mathbb{C}.$$

Non-Coupled Case

Writing out the full Schrödinger equation in the non-coupled case results in

$$0 = \left(\frac{-1}{m} + \frac{C_F\alpha_s\mu}{\pi m^2} + \frac{e}{rm^2}\right)\partial_r^2 A_{J,0;s;l}(r) + \left(\frac{-2}{mr} + \frac{-e}{r^2m^2} + \frac{C_F\alpha_s\mu}{2\pi m^2}\frac{4}{r} + \frac{2e}{r^2m^2}\right)\partial_r A_{J,0;s;l}(r) + \left(\frac{l(l+1)}{r^2} + \left(\frac{-e}{r} + \sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right)\right.$$
$$\left. + \frac{1}{m^2}\left(2l(l+1)\left(\frac{-e}{4r^3} - \frac{\sigma}{12r} - \frac{C_F\alpha_s\mu}{2\pi r^2}\right) + \left(\frac{5e}{2r^3} + \frac{\sigma}{2r}\right)\frac{J(J+1) - l(l+1) - s(s+1)}{2} + \frac{\sqrt{2\pi}e}{r^3}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right) - E\right) A_{J,0;s;l}(r).$$

Assuming that the asymptotic solution arises by neglecting all but the dominant terms leads to

$$0 = \frac{e}{rm^2}\partial_r^2 A_{J,0;s;l}^{r\to 0}(r) + \frac{e}{r^2m^2}\partial_r A_{J,0;s;l}^{r\to 0}(r) + \frac{1}{m^2}\left(l(l+1)\frac{-e}{2r^3} + \frac{5e(J(J+1) - l(l+1) - s(s+1))}{4r^3} + \frac{\sqrt{2\pi}e}{r^3}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right) A_{J,0;s;l}^{r\to 0}(r)$$

or

$$0 = r^2\partial_r^2 A_{J,0;s;l}^{r\to 0}(r) + r\partial_r A_{J,0;s;l}^{r\to 0}(r) + \left(\frac{-l(l+1)}{2} + \frac{5(J(J+1) - l(l+1) - s(s+1))}{4} + \sqrt{2\pi}\langle J,0;s;l|\mathcal{CT}|J,0;s;l\rangle\right) A_{J,0;s;l}^{r\to 0}(r).$$

The coefficient of $A_{J,0;s;l}^{r\to 0}(r)$ is from now on abbreviated with C_{SQP}^{NCC} (**s**tatic **q**uark **p**otential, **n**on-**c**oupled **c**ase).

This is yet again a Cauchy-Euler differential equation and thus has the two independent solutions

$$A_{J,0;s;l}^{r\to 0}(r) = r^{\lambda_\pm}$$

for $\lambda_\pm$ as the roots of

$$x^2 + x + C_{SQP}^{NCC},$$

i.e.

$$\lambda_\pm = \pm\sqrt{-C_{SQP}^{NCC}}.$$

The final solution must satisfy the assumption, i.e.

$$0 = \left(\frac{-r^2}{m} + \frac{C_F\alpha_s\mu r^2}{\pi m^2}\right)\partial_r^2 A_{J,0;s;l}^{r\to 0}(r) + \left(\frac{-2r}{m} + 4\frac{C_F\alpha_s\mu r}{2\pi m^2}\right)\partial_r A_{J,0;s;l}^{r\to 0}(r) + \left(l(l+1) + \left(-er + \sigma r^3\right) + \frac{1}{m}\left(\frac{-8e^2}{9} + \frac{2\sigma}{\pi}r^2\ln(r)\right)\right.$$
$$\left. + \frac{1}{m^2}\left(2l(l+1)\left(-\frac{\sigma r}{12} - \frac{C_F\alpha_s\mu}{2\pi}\right) + \frac{\sigma r}{2}\frac{J(J+1) - l(l+1) - s(s+1)}{2}\right) - Er^2\right) A_{J,0;s;l}^{r\to 0}(r) \qquad \text{for } r \to 0.$$

It is apparent that this assumption only holds if and only if

$$\lim_{r\to 0} r^{\lambda_\pm} = 0.$$

This is the case for precisely λ_+ since $\lambda_- < 0$ (it will be verified numerically that $C_{SQP}^{NCC} < 0$). Consequently, the total solution is

$$A_{J,0;s;l}^{r\to 0}(r) = B \cdot r^{\lambda_+} \qquad \text{with } \lambda_+ = \sqrt{-C_{SQP}^{NCC}} \qquad \text{and } B \in \mathbb{C}.$$

Coupled Case

The general Schrödinger equation in the coupled case is

$$0 = \left(\frac{-1}{m} + \frac{C_F\alpha_s\mu}{\pi m^2} + \frac{e}{rm^2}\right)\partial_r^2 A_{J,0;s;l}(r) + \left(\frac{-2}{mr} + \frac{-e}{r^2m^2} + \frac{C_F\alpha_s\mu}{2\pi m^2}\frac{4}{r} + \frac{2e}{r^2m^2}\right)\partial_r A_{J,0;s;l}(r) + \left(\frac{1}{r^2}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix} + \left(\frac{-e}{r}+\sigma r\right) + \frac{1}{m}\left(\frac{-8e^2}{9r^2} + \frac{2\sigma}{\pi}\ln(r)\right)\right.$$
$$+ \frac{1}{m^2}\left(2\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix}\left(\frac{-e}{4r^3} - \frac{\sigma}{12r} - \frac{C_F\alpha_s\mu}{2\pi r^2}\right) + \left(\frac{5e}{2r^3} + \frac{\sigma}{2r}\right)\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix}\right.$$
$$\left.\left. + \frac{\sqrt{2\pi}e}{r^3}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right) - E\right)\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}.$$

Due to the assumption all non-dominating terms can be eliminated, resulting in

$$0 = \frac{e}{rm^2}\partial_r^2 A_{J,0;s;l}(r) + \frac{e}{r^2m^2}\partial_r A_{J,0;s;l}(r) + \frac{1}{m^2}\left(\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix}\frac{-e}{2} + \frac{5e}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix}\right.$$
$$\left. + \sqrt{2\pi}e\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right)\frac{1}{r^3}\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}$$

or

$$0 = r^2\begin{pmatrix}\partial_r^2 A_{J,0;1;J-1}(r) \\ \partial_r^2 A_{J,0;1;J+1}(r)\end{pmatrix} + r\begin{pmatrix}\partial_r A_{J,0;1;J-1}(r) \\ \partial_r A_{J,0;1;J+1}(r)\end{pmatrix} + \left(\frac{-1}{2}\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix} + \frac{5}{2}\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix}\right.$$
$$\left. + \sqrt{2\pi}\begin{pmatrix}\langle J,0;1;J-1|\mathcal{CT}|J,0;1;J-1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J-1\rangle \\ \langle J,0;1;J-1|\mathcal{CT}|J,0;1;J+1\rangle & \langle J,0;1;J+1|\mathcal{CT}|J,0;1;J+1\rangle\end{pmatrix}\right)\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}.$$

As in all the previous cases, the coefficient of $(A_{J,0;1;J-1}(r), A_{J,0;1;J+1}(r))$ will be abbreviated with C_{SQP}^{CC} (**s**tatic **q**uark **p**otential, **c**oupled **c**ase).

Motivated by the one method of solving the Cauchy-Euler differential equation, the substitution is carried out as

$$v = \ln(r) \qquad \begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix} = u(v) = \begin{pmatrix} u_-(v) \\ u_+(v)\end{pmatrix},$$

leading to

$$0 = \partial_v^2 u + C_{SQP}^{CC}u(v).$$

This is a linear differential equation and can easily be rewritten into a first order multi-dimensional differential equation.

First, consider the vector-valued function $w(v)$ together with the matrix A:

$$w(v) = \begin{pmatrix} u_-(v) \\ u_+(v) \\ \partial_v u_-(v) \\ \partial_v u_+(v)\end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -C_{SQP}^{CC} & & 0 & 0 \\ & 0 & 0 & 0\end{pmatrix}$$

Next, the above linear differential equation is equivalent to

$$Aw(v) = \partial_v w.$$

Such differential equations are well-known to have the solution

$$w(v) = \exp(Av)w(0).$$

As a consequence, this means that any solution

$$\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}$$

of the original asymptotic Schrödinger equation is a linear superposition of

$$\begin{pmatrix} (\exp(A\ln(r))e_i)_1 \\ (\exp(A\ln(r))e_i)_2\end{pmatrix},$$

where e_i enumerates the standard basis vectors of $\mathbb{C}^4$. Here only the first two components of the solution $w(v) = \exp(Av)e_i$ are being considered since the last two are simply the derivatives of the first.

Now, all that remains is to verify the assumption. Here the assumption works out to be

$$0 = \left(\frac{-r^2}{m} + \frac{C_F\alpha_s\mu r^2}{\pi m^2}\right)\partial_r^2 A_{J,0;s;l}(r) + \left(\frac{-2r}{m} + 4\frac{C_F\alpha_s\mu r}{2\pi m^2}\right)\partial_r A_{J,0;s;l}(r) + \left(\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix} + (-er+\sigma r^3) + \frac{1}{m}\left(\frac{-8\alpha^2}{9} + \frac{2\sigma}{\pi}r^2\ln(r)\right)\right.$$
$$\left. + \frac{1}{m^2}\left(2\begin{pmatrix} J\cdot(J-1) & 0 \\ 0 & (J+1)(J+2)\end{pmatrix}\left(-\frac{\sigma r}{12} - \frac{C_F\alpha_s\mu}{2\pi}\right) + \left(\frac{\sigma r}{2}\right)\begin{pmatrix} J-1 & 0 \\ 0 & -(J+1)\end{pmatrix}\right) - Er^2\right)\begin{pmatrix} A_{J,0;1;J-1}(r) \\ A_{J,0;1;J+1}(r)\end{pmatrix}.$$

This condition can now be verified numerically to find the asymptotic solutions.

Chapter 4

Numerics and Results

In this chapter, the numerical methods used to compute the explicit energy levels of bottomonium are explained and the resulting energy levels are discussed.

Numerical Calculations

Before it is possible to come to the numerical computations, the unknown additive offset present in the static quark potential needs to be addressed. Regarding the additive offset, there are no concrete values. Therefore, the energy levels of some base state, in this case 0^{-+}, were first computed while neglecting the additive offset and later compared with their experimental counterpart. In the present situation, the additive offset was then chosen to be the difference between the computed first energy level of 0^{-+} and the experimental value of $9398.7\,\text{MeV}$. This additive offset was in turn incorperated into all subsequent calculations. It should be noted that the offset was recalculated for each of the five main corrections. The second unknown constant is the string tension σ. While there do exist some concrete values for σ, the author chose to use the correction with the full static quark potential to match σ such that the energy difference between the first and second energy level of 0^{+-} is realized. This works out to a value of $\sigma = 0.282159\,\text{GeV}^2$ which is used for all other cases.

In order to compute the energy levels in the coupled case there is one very useful trick from [5] which greatly reduces the complexity of the calculations. To see where this trick is coming from, observe that the solution A to the coupled differential equation is a linear combination of the two full solutions A_1 and A_2 obtained by using the two different asymptotic behaviors found for the differential equation. In particular, this means that if a specific energy E is in fact an energy level of bottomonium, then two constants C_1 and C_2 must exist such that

$$0 = A(\infty) = C_1 A_1(\infty) + C_2 A_2(\infty) = \begin{pmatrix} A_1(\infty) & A_2(\infty) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This implies that the matrix $(A_1(\infty)\ A_2(\infty))$, recall that $A_i(\infty)$ is a two-dimensional vector, has a non-trivial kernel and thus has a determinant of zero. Furthermore, this entails that if the precise value of the constants C_1 and C_2 is irrelevant, then it fully suffices to compute $A_1(\infty)$ and $A_2(\infty)$ and check that the determinant of the combined matrix is zero.

In order to actually solve the occurring differential equations, a Runge-Kutta ordinary differential equation solver, namely the Dormand–Prince (RKDP) method which is also implemented in the boost C++ library, was applied. For all corrections with the exception of the ultimate one involving the complete static quark potential an adaptive step size could be used inside the Runge-Kutta algorithm. In the case containing the full static quark potential, this was sadly not possible in a reasonable amount of steps, and so it was elected to use a fixed step size for these calculations. More precisely, first a starting energy E was specified and substituted into the differential equation derived in chapter 2. Then the asymptotic solution as well as its derivative were evaluated at the set start point of the Runge-Kutta algorithm after which the algorithm was run. It was then tested if the second boundary condition, namely that the solution drops of to zero at infinity was satisfied. If this did not happen, the energy E could be ruled out as a viable energy level of the considered state. Specifically, the locating of the sought after energy levels was accomplished by first conducting a scan after which possible candidates where identified via a sign change, and it was subsequently possible to find the precise energy levels by conducting a bilinear search.

Results

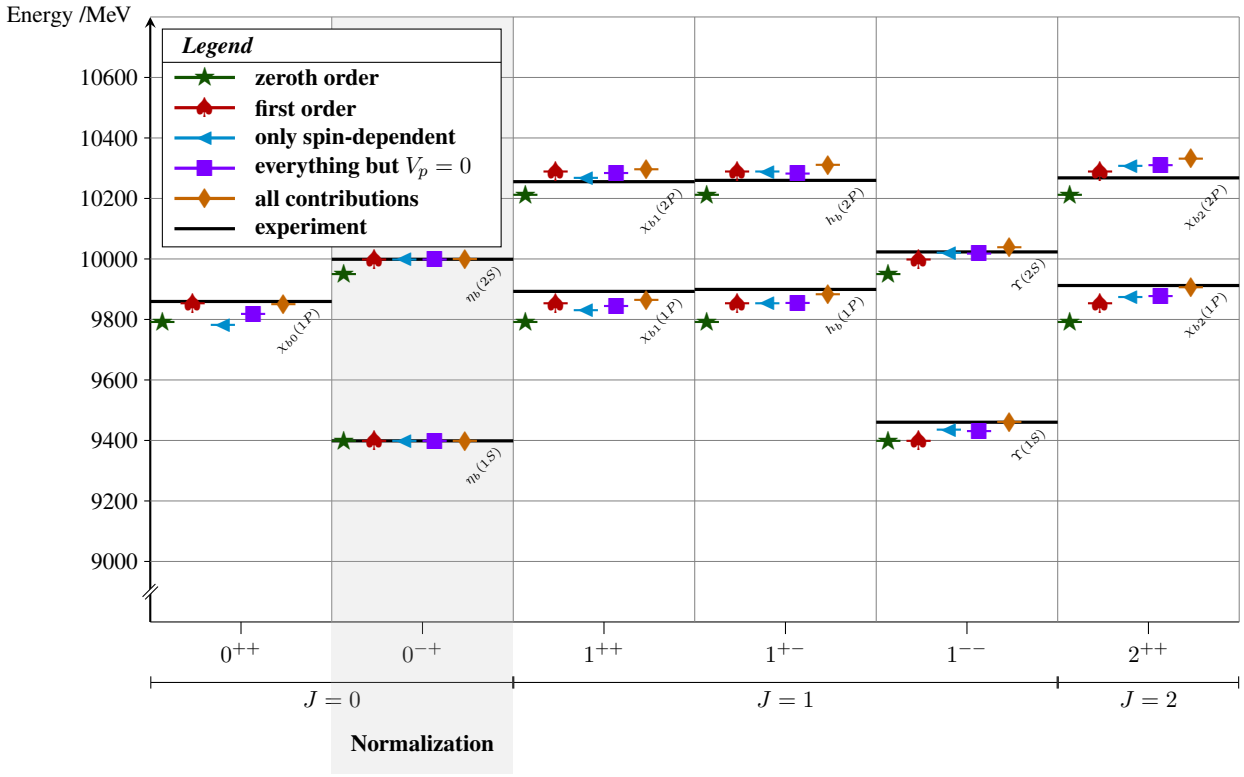

Figure 4.1: The computed energies of the different states of bottomonium in the different cases are compared with the experimental values which stem from [12]. The experimentally obtained energies are represented by thick black lines.

The computed results are shown in Figure 4.1 and the exact values can be found in Appendix B. To validate these results, it might help to make some elementary observations. First of all, it is self-explanatory that the zeroth and first order corrections depend only on the angular momentum l. This means that the energy levels should be the same for $0^{++}, 1^{++}$, and 1^{+-}. Evidently, this is here the case. Secondly, the results get better with the additions of further corrections. To analyze this more qualitatively, consider the norm

$$\sum_{\substack{C \text{ computed value} \\ E \text{ experimental value}}} \frac{(E-C)^2}{E^2},$$

where the sum is taken over all computed energy levels with respectable experimental counterparts, i.e. in this case the first energy level of 0^{++} as well as the first and second energy levels of $0^{-+}, 0^{++}, 1^{++}, 1^{+-}, 1^{--}, 2^{++}$ were selected. The resulting norms are shown in Figure 4.2.

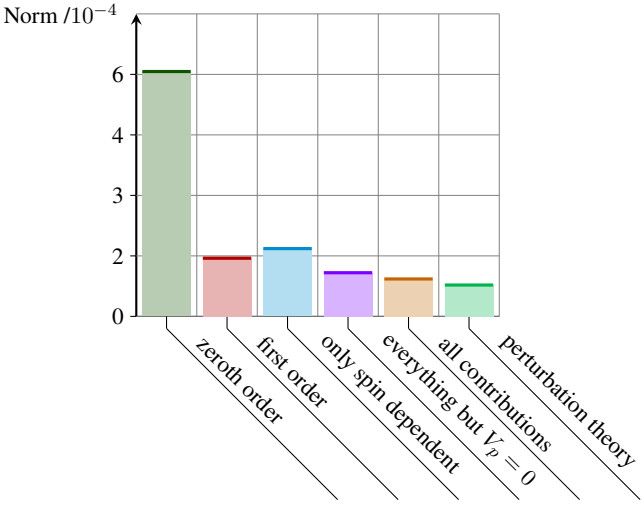

Figure 4.2: For the different cases in this thesis, the values of the norm are compared with each other as well as with the results obtained by using perturbation theory in [11] by Michael Eichberg and Marc Wagner.

The first thing to notice is that there is a massive increase in accuracy when transitioning from the zeroth order to the first order correction. The resulting energy values are actually slightly better than the consecutive corrections involving the spin-dependent contributions but are worse than those given by the complete static quark potential with $V_{\mathrm{p}}(r)$ being set to zero. Since the latter of which involves the same asymptotic solutions with a constant change, it is always better to use the full static quark potential with $V_{\mathrm{p}}(r)$ instead of only the

spin-dependent corrections. If even more accuracy is required, the full static quark potential can be used, but it should be noted that it needs substantially more computational resources due to the occurrence of higher orders of r in front of the differentials. Lastly, observe that all corrections struggle with accurately predicting the second energy level of 2^{++}. This behavior will be addressed in the conclusion.

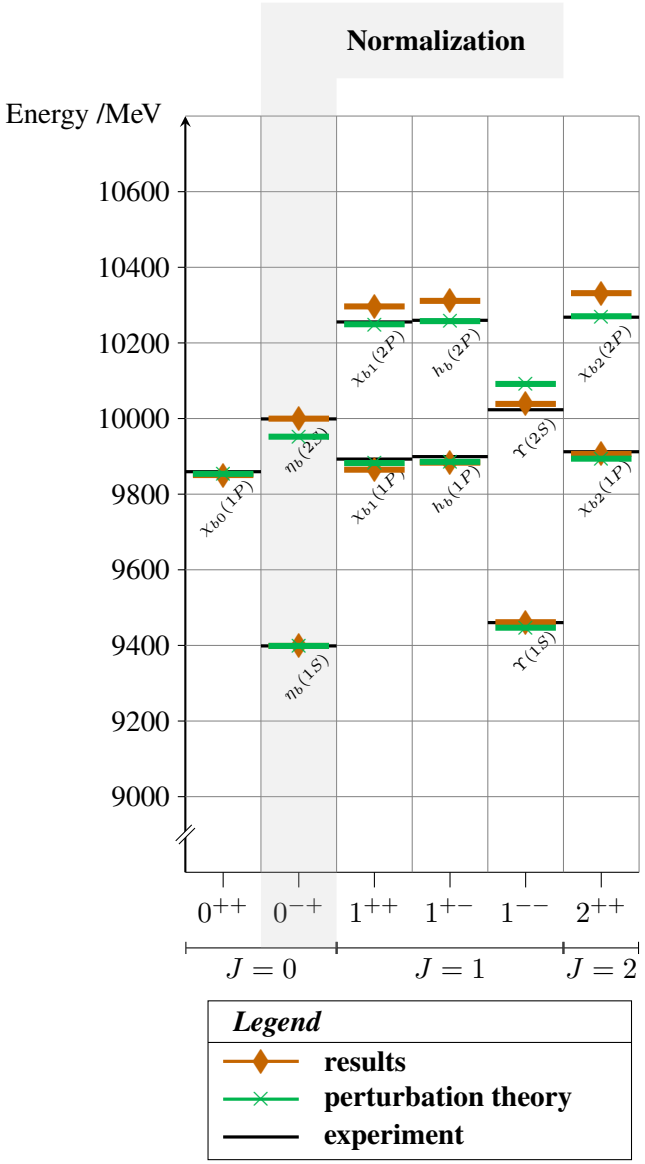

Figure 4.3: The results computed throughout this thesis are compared with those presented by Michael Eichberg and Marc Wagner in [11].

As already mentioned in the introduction, similar computations to those presented in this thesis have already been conducted by Michael Eichberg and Marc Wagner in [11]. In their paper, they used time-independent perturbation theory to arrive at their energy levels. It should be noted that their computations used a set string tension σ of $0.23\,\text{GeV}^2$. When compared with the full static quark potential computation in this thesis, their values are objectively better, see Figure 4.2. A more complete picture can be seen in Figure 4.3. In the state 0^{++}, both the presented approach from this thesis as well as the perturbative one lead to very good approximations of the energy levels. Regarding 0^{-+}, it appears as if the here presented results are more accurate, but this is deceiving since the string tension σ was explicitly chosen in this way. Concerning the states 1^{-+} and 1^{++}, the perturbative computations are unilaterally better and the same can also be said for 2^{++}.

Chapter 5

Conclusion

In this thesis, it was possible to compute the energy levels of different bottomonium bounded states, starting from the static quark potential. In doing so, a method for finding the asymptotic solutions of a variety of differential equations was introduced.

Using the full static quark potential, it is within reach to make predictions regarding the energy levels of other states of bottomonium. Consider for instance the second energy level of the state 0^{++}. The computed value works out to be $10293.6\,\text{MeV}$ whereas experimentally a particle with a mass of $10232.5\,\text{MeV}$ (see [12]) was found which is presumed to correspond to this energy level. The similarity of the experimental data and of the here computed value now provide additional encouragements that this is indeed the case. Similar can be said for the first energy level $\Upsilon_2(1D)$ of 2^{--} where experimentally masses of $10163.7\,\text{MeV}$ (see [12]) come into question while computationally the mass of the state is $10165.8\,\text{MeV}$.

Despite these advances, there are some improvements that can be implemented. One major deficit of the employed numerical techniques lie in the computation of the asymptotic conditions in the coupled case. Here infinite sums occur. Although these do converge, the convergence is sometimes quite slow, leading to very large summands. Computationally, these summands get so large that they cause large inaccuracies in the final computation of the asymptotic approximation of the solutions. These numerical inaccuracies can be alleviated by computing the large summands not in double precision but rather keeping all digits pertained to the integer part of the resulting values and choosing a fixed number of decimal points. This ought to hopefully allow for a decrease in the start point of the Runge-Kutta algorithm for these coupled cases and thus lead to an increase in precision of the energy levels. Here it should be pointed out that the computed energy levels appear somewhat stable against inaccuracies in the asymptotic behavior around 0. This might also be a reason for why the two different asymptotic behaviors derived in the correction, where only spin-dependent components are considered, do not dramatically impact the final energy levels. It would be ideal if it were possible to completely eliminate one of these asymptotic behaviors entirely in the future. Furthermore, the string tension is currently being matched with experimental energy values. It seems reasonable that computing the string tension with methods in lattice QCD could render this unnecessary.

In the future, it is also desirable to consider resonances as seen in [9]. This means that the bottomonium will be free to decay into a pair of mesons. This can occur if the energy between the bottom quark and its antiparticle is so large that it suffices to create a new light quark and its antiparticle. These can then pair with the bottom quark and antiquark, respectively, to form two heavy-light mesons.

Appendix A

Derivation of the Explicit Radial Potentials in the Static Quark Potential

The aim of this appendix is to provide a precise justification of the individual components of the radial potentials $V_{\mathrm{LS}}(r)$, $V_{S_{12}}(r)$, $V_{S^2}(r)$, $V_{\mathrm{r}}(r)$, $V_{\mathrm{L}}(r)$, and $V_{\mathrm{p}}(r)$:

Explicit Forms for the Radial Potentials (due to [1], [6], [10])

$$V_{\mathrm{LS}}(r) = \frac{5e}{2r^3} + \frac{\sigma}{2r} \qquad V_{\mathrm{r}}(r) = \left(\frac{3}{4} + d_s\right)\frac{\delta(r)}{r^2}$$

$$V_{S_{12}}(r) = \frac{3e}{r^3} \qquad V_{\mathrm{L}}(r) = \frac{e}{4r^3} - \frac{\sigma}{12r}$$

$$V_{S^2}(r) = \frac{2e\delta(r)}{r^2} \qquad V_{\mathrm{p}}(r) = -\frac{C_F\alpha_s\mu}{2\pi} - \frac{e}{2r}$$

The potentials $V_{S^2}(r)$ and $V_{\mathrm{r}}(r)$ are only shown for completeness. They will not be used throughout this thesis.

$$m_b = 4.977 \text{ GeV} \qquad \sigma = 0.282159 \text{ GeV}^2$$

$$\alpha_s = 0.2815 \qquad \mu = 1.5879557\frac{\pi}{a} \qquad \text{(see [1, p. 37])}$$

$$C_F = 4/3 \qquad e = C_F\alpha_s \qquad \text{(see [1, p. 97])}$$

The string tension σ was determined by comparing certain computed energies with their experimental counterpart. The details of this will be discussed in chapter 4.

In his master thesis, Michael Eichberg used the following expression for the spin-dependent contribution to the static quark potential

$$H_{\mathrm{SD}}(\mathbf{r}) = \left(\frac{\mathbf{L}\mathbf{S}_1}{m_1^2} + \frac{\mathbf{L}\mathbf{S}_2}{m_2^2}\right)\frac{V_0'(r) + 2V_2'(r)}{2r} + \frac{\mathbf{L}(\mathbf{S}_1 + \mathbf{S}_2)}{m_1 m_2}\frac{V_2'(r)}{r} + \left(\frac{(\mathbf{S}_1\mathbf{r})(\mathbf{S}_2\mathbf{r})}{m_1 m_2} - \frac{\mathbf{S}_1\mathbf{S}_2}{3m_1 m_2}\right)V_3(r) + \frac{\mathbf{S}_1\mathbf{S}_2}{m_1 m_2}V_4(r)$$

(see [10, (2.11)]) together with the potentials

$$V_0(r) = -\frac{e}{r} + \sigma r, \quad \rightsquigarrow \quad V_0'(r) = \frac{e}{r^2} + \sigma,$$

$$V_1'(r) = -\sigma, \qquad V_3(r) = \frac{3e}{r^3},$$

$$V_2'(r) = \frac{e}{r^2}, \qquad V_4(r) = 8\pi e\delta^3(\mathbf{r})$$

(see [1, (6.80), (6.83)-(6.86)] with $h(\mu) = 0$). The above formula is of course more general than necessary since it governs the interaction of a quark-antiquark pair with possibly unequal masses. In the situation of this thesis, only quark-antiquark pairs possessing equal masses will be considered and therefore $m_1 = m_2 = m$.

This leads to the following simplified form for the spin-dependent part of the static quark potential

$$\begin{aligned} H_{\mathrm{SD}}(\mathbf{r}) =& \frac{\mathbf{LS}}{m^2}\left(\frac{V_0'(r)+2V_2'(r)}{2r}+\frac{V_2'(r)}{r}\right)+\left(\frac{(\mathbf{S_1r})(\mathbf{S_2r})}{m^2}-\frac{\mathbf{S_1S_2}}{3m^2}\right)V_3(r)+\frac{\mathbf{S_1S_2}}{m^2}V_4(r) \\ =& \frac{1}{m^2}\left(\left(\frac{V_0'(r)}{2r}+\frac{2V_2'(r)}{r}\right)\mathbf{LS}+V_3(r)\mathbf{S}_{12}(\mathbf{r})+V_4(r)(\mathbf{S_1S_2})\right). \end{aligned}$$

A comparison with the presented spin-dependent contribution of the static quark potential yields

$$\begin{aligned} V_{\mathrm{LS}}(r) =& \frac{V_0'(r)}{2r}+\frac{2V_2'(r)}{r} & V_{S_{12}}(r) =& V_3(r)=\frac{3e}{r^3}, \\ =& \frac{e}{2r^3}+\frac{\sigma}{2r}+\frac{2e}{r^3} & V_{S^2}(r) =& V_4(r)=8\pi e\delta^3(\mathbf{r}) \\ =& \frac{5e}{2r^3}+\frac{\sigma}{2r}, & =& \frac{2e\delta(r)}{r^2}. \end{aligned}$$

Here the identity $\delta^3(\mathbf{r})=\frac{\delta(r)}{4\pi r^2}$, which can easily be verified by observing that

$$\begin{aligned} \int d\mathbf{r}^3\, f(\mathbf{r})\frac{\delta(r)}{4\pi r^2} &= \int_0^\infty dr\, r^2\int_{-\pi}^{\pi}d\varphi\int_0^{\pi}d\theta\,\sin(\theta)f(r\cos\varphi\sin\theta,r\sin\varphi\sin\theta,r\cos\theta)\frac{\delta(r)}{4\pi r^2} \\ &= \frac{f(0,0,0)}{4\pi}\int_{-\pi}^{\pi}d\varphi\int_0^{\pi}d\,\theta\sin(\theta)=f(0)=\int d\mathbf{r}^3\, f(\mathbf{r})\delta^3(r), \end{aligned}$$

was used.

Now, it is time to study the spin-independent contributions. These are slightly more involved in the sense that the spin-independent part of the Hamiltonian

$$\begin{aligned} H_{\mathrm{SI}} =& \frac{1}{m_1^2}\left(\frac{1}{2}\left[\mathbf{p}_1^2,V_{\mathbf{p}^2}^{(2,0)}(r)\right]_+ + V_r^{(2,0)}(r)+V_{\mathbf{L}^2}^{(2,0)}(r)\frac{\mathbf{L}_1^2}{r^2}\right) \\ &+\frac{1}{m_2^2}\left(\frac{1}{2}\left[\mathbf{p}_2^2,V_{\mathbf{p}^2}^{(0,2)}(r)\right]_+ + V_r^{(0,2)}(r)+V_{\mathbf{L}^2}^{(0,2)}(r)\frac{\mathbf{L}_2^2}{r^2}\right) \\ &+\frac{1}{m_1m_2}\left(-\frac{1}{2}\left[\mathbf{p}_1\mathbf{p}_2,V_{\mathbf{p}^2}^{(1,1)}(r)\right]_+ + V_r^{(1,1)}(r)-V_{\mathbf{L}^2}^{(1,1)}(r)\frac{\mathbf{L}_1\mathbf{L}_2+\mathbf{L}_2\mathbf{L}_1}{2r^2}\right) \end{aligned}$$

(see [6, (3)-(5)], again allowing for unequal masses) is expressed in terms of $V_{\mathbf{p}^2}^{(2,0)}(r), V_{\mathbf{p}^2}^{(1,1)}(r), V_{\mathbf{p}^2}^{(0,2)}(r)$, $V_{\mathbf{L}^2}^{(2,0)}(r), V_{\mathbf{L}^2}^{(1,1)}(r), V_{\mathbf{L}^2}^{(0,2)}(r)$, and $V_r^{(2,0)}, V_r^{(1,1)}, V_r^{(0,2)}$ which in turn are linked via the equations

$$\begin{aligned} V_b(r) =& -\frac{2}{3}V_{\mathbf{L}^2}^{(1,1)}(r)-V_{\mathbf{p}^2}^{(1,1)}(r), & V_d(r) =& \frac{2}{3}V_{\mathbf{L}^2}^{(2,0)}(r)+V_{\mathbf{p}^2}^{(2,0)}(r), \\ V_c(r) =& -V_{\mathbf{L}^2}^{(1,1)}(r), & V_e(r) =& V_{\mathbf{L}^2}^{(2,0)}(r) \end{aligned}$$

(see [6, (76)-(79)] and note that there E is used instead of V to denote the potentials) to the commonly used potentials

$$\begin{aligned} V_,C_b + \frac{2}{3}\frac{e}{r}-\frac{\sigma}{9}r, & & V_d =& C_d-\frac{\sigma}{9}r, \\ V_c =& -\frac{1}{2}\frac{e}{r}-\frac{\sigma}{6}r, & V_e =& -\frac{\sigma}{6}r \end{aligned}$$

(see [1, (6.88) - (6.91)]). In the present specialized situation, of a quark and its antiquark interacting, $m_1=m_2=m$, $\mathbf{p}=\mathbf{p}_1=-\mathbf{p}_2$, $\mathbf{L}=\mathbf{L}_1=-\mathbf{L}_2$, $V_{\mathbf{p}^2}^{(2,0)}(r)=V_{\mathbf{p}^2}^{(0,2)}(r)$, $V_{\mathbf{L}^2}^{(2,0)}(r)=V_{\mathbf{L}^2}^{(0,2)}(r)$, $V_r^{(2,0)}=V_r^{(0,2)}$, and therefore the Hamiltonian reduces to

$$H_{\mathrm{SI}} =\frac{1}{m^2}\left(\frac{1}{2}\left[\mathbf{p}^2,2V_{\mathbf{p}^2}^{(2,0)}(r)+V_{\mathbf{p}^2}^{(1,1)}(r)\right]_+ + 2V_r^{(2,0)}(r)+V_r^{(1,1)}(r)+(2V_{\mathbf{L}^2}^{(2,0)}(r)+V_{\mathbf{L}^2}^{(1,1)}(r))\frac{\mathbf{L}^2}{r^2}\right).$$

At last, comparing coefficients with the presented static quark potential results in

$$\begin{aligned} V_{\mathrm{p}}(r) =& V_{\mathbf{p}^2}^{(2,0)}(r)+\frac{1}{2}V_{\mathbf{p}^2}^{(1,1)}(r) \\ =& \left(V_d(r)-\frac{2}{3}V_{\mathbf{L}^2}^{(2,0)}(r)\right)+\frac{1}{2}\left(-V_b(r)-\frac{2}{3}V_{\mathbf{L}^2}^{(1,1)}(r)\right) \\ =& \left(V_d(r)-\frac{2}{3}V_e(r)\right)+\frac{1}{2}\left(-V_b(r)+\frac{2}{3}V_c(r)\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(C_d - \frac{\sigma}{9}r - \frac{2}{3}\left(-\frac{\sigma}{6}r\right)\right) + \frac{1}{2}\left(-C_b - \frac{2}{3}\frac{e}{r} + \frac{\sigma}{9}r + \frac{2}{3}\left(-\frac{1}{2}\frac{e}{r} - \frac{\sigma}{6}r\right)\right) \\
&= C_d - \frac{C_b}{2} - \frac{e}{2r},
\end{aligned}
$$

$$
\begin{aligned}
V_{\mathrm{L}}(r) &= \frac{1}{2r^2}\left(2V_{\mathbf{L}^2}^{(2,0)}(r) + V_{\mathbf{L}^2}^{(1,1)}(r)\right) \\
&= \frac{1}{2r^2}\left(2V_e(r) - V_c(r)\right) \\
&= \frac{1}{2r^2}\left(2\left(-\frac{\sigma}{6}r\right) + \frac{1}{2}\frac{e}{r} + \frac{\sigma}{6}r\right) \\
&= \frac{e}{4r^3} - \frac{\sigma}{12r}.
\end{aligned}
$$

For the purposes of this thesis, the values

$$C_b = 0 \qquad \text{and} \qquad C_d = -\frac{1}{4}V_{\text{self}}(\mu) = -\frac{C_F\alpha_s\mu}{2\pi}$$

(compare [1, (4.46), (6.92)]) will be used which yield the final forms for $V_{\mathrm{p}}(r)$ and $V_{\mathrm{L}}(r)$.

The last remaining potential $V_{\mathrm{r}}(r)$ can be obtained by comparing coefficients of the presented static quark potential with the form given in [1, (6.95)-(6.100)]:

$$
\begin{aligned}
V_{\mathrm{r}}(r) &= \left(\frac{3}{4} + d_s\right)4\pi\delta^3(r) \\
&= \left(\frac{3}{4} + d_s\right)\frac{\delta(r)}{r^2}
\end{aligned}
$$

Appendix B

Computed Energies and Matrix Elements

State	Label J^{PC}	Experimental Mass MeV	Computed Mass (Normed) MeV	Relative Error %
Zeroth Order				
$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9791.69	0.687176
$\eta_b(1S)$	0^{-+}	9398.7 ± 2.0	9398.7	0
$\eta_b(2S)$		9999 ± 4	9950.02	0.489865
$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9791.69	1.02187
$\chi_{b1}(2P)$		10255.46 ± 0.50	10212.1	0.422338
$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9791.69	1.08706
$h_b(2P)$		10259.8 ± 1.2	10212.1	0.464461
$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9398.7	0.651142
$\Upsilon(2S)$		10023.26 ± 0.31	9950.02	0.730717
$\chi_{b2}(1P)$	2^{++}	$9912.21 \pm 0.26 \pm 0.31$	9791.69	1.21589
$\chi_{b2}(2P)$		10268.65 ± 0.5	10212.1	0.550245
First Order				
$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9853.49	0.0603974
$\eta_b(1S)$	0^{-+}	9398.7 ± 2.0	9398.7	0
$\eta_b(2S)$		9999 ± 4	9998.21	0.0078836
$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9853.49	0.397207
$\chi_{b1}(2P)$		10255.46 ± 0.50	10289.1	0.32831
$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9853.49	0.462809
$h_b(2P)$		10259.8 ± 1.2	10289.1	0.28587
$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9398.7	0.651142
$\Upsilon(2S)$		10023.26 ± 0.31	9998.21	0.249902
$\chi_{b2}(1P)$	2^{++}	$9912.21 \pm 0.26 \pm 0.31$	9853.49	0.59245
$\chi_{b2}(2P)$		10268.65 ± 0.5	10289.1	0.199439
Only Spin-Dependent Contribution				
$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9782.11	0.784343
$\eta_b(1S)$	0^{-+}	9398.7 ± 2.0	9398.7	0
$\eta_b(2S)$		9999 ± 4	9998.21	0.0078836
$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9830.65	0.628002
$\chi_{b1}(2P)$		10255.46 ± 0.50	10268.1	0.122817

State	Label J^{PC}	Experimental Mass MeV	Computed Mass (Normed) MeV	Relative Error %
$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9853.49	0.462809
$h_b(2P)$		10259.8 ± 1.2	10289.1	0.28587
$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9435.77	0.259291
$\Upsilon(2S)$		10023.26 ± 0.31	10020.7	0.0260083
$\chi_{b2}(1P)$	2^{++}	$9912.21 \pm 0.26 \pm 0.31$	9874.17	0.383782
$\chi_{b2}(2P)$		10268.65 ± 0.5	10307.8	0.380797
Complete Potential with $V_p = 0$				
$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9818.34	0.416873
$\eta_b(1S)$	0^{-+}	9398.7 ± 2.0	9398.7	0
$\eta_b(2S)$		9999 ± 4	9998.21	0.0078836
$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9844.48	0.488222
$\chi_{b1}(2P)$		10255.46 ± 0.50	10284.1	0.279079
$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9854.58	0.45176
$h_b(2P)$		10259.8 ± 1.2	10282.3	0.219718
$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9430.95	0.310285
$\Upsilon(2S)$		10023.26 ± 0.31	10017.7	0.0551397
$\chi_{b2}(1P)$	2^{++}	$9912.21 \pm 0.26 \pm 0.31$	9877.32	0.352159
$\chi_{b2}(2P)$		10268.65 ± 0.5	10310.1	0.403336
Complete Potential				
$\chi_{b0}(1P)$	0^{++}	$9859.44 \pm 0.42 \pm 0.31$	9850.54	0.09031
$\eta_b(1S)$	0^{-+}	9398.7 ± 2.0	9398.7	0
$\eta_b(2S)$		9999 ± 4	9999.74	0.00744996
$\chi_{b1}(1P)$	1^{++}	$9892.78 \pm 0.26 \pm 0.31$	9864.74	0.283389
$\chi_{b1}(2P)$		10255.46 ± 0.50	10296.5	0.400585
$h_b(1P)$	1^{+-}	9899.3 ± 0.8	9883.35	0.161138
$h_b(2P)$		10259.8 ± 1.2	10311.2	0.500509
$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26	9461.38	0.0113715
$\Upsilon(2S)$		10023.26 ± 0.31	10038.7	0.154431
$\chi_{b2}(1P)$	2^{++}	$9912.21 \pm 0.26 \pm 0.31$	9906.48	0.0577752
$\chi_{b2}(2P)$		10268.65 ± 0.5	10331.6	0.61313

Table B.1: The computed energies for the different states of bottomonium are shown. The results are split up with regard to the different cases considered in this thesis. The grayed out values were applied to normalize the static quark potential and thus cannot be used to draw conclusions. The experimental values stem from [12].

State	$\mathcal{CT}$	0^{++}	0^{-+}	1^{++}	1^{+-}	1^{--}		2^{++}		2^{-+}	2^{--}
		$\lvert 0,0;1;1\rangle$	$\lvert 0,0;0;0\rangle$	$\lvert 1,0;1;1\rangle$	$\lvert 1,0;0;1\rangle$	$\lvert 1,0;1;0\rangle$	$\lvert 1,0;1;2\rangle$	$\lvert 2,0;1;1\rangle$	$\lvert 1,0;1;3\rangle$	$\lvert 2,0;0;2\rangle$	$\lvert 2,0;1;2\rangle$
0^{++}	$\langle 0,0;1;1\rvert$	$-\frac{1}{\sqrt{2\pi}}$									
0^{-+}	$\langle 0,0;0;0\rvert$	0									
1^{++}	$\langle 1,0;1;1\rvert$			$\frac{1}{\sqrt{8\pi}}$							
1^{+-}	$\langle 1,0;0;1\rvert$				0						
1^{--}	$\langle 1,0;1;0\rvert$					0	$\frac{1}{2\sqrt{\pi}}$				
	$\langle 1,0;1;2\rvert$					$\frac{1}{2\sqrt{\pi}}$	$-\frac{1}{\sqrt{8\pi}}$				
2^{++}	$\langle 2,0;1;1\rvert$							$-\frac{1}{10\sqrt{2\pi}}$	$\frac{3}{10}\sqrt{\frac{3}{\pi}}$		
	$\langle 1,0;1;3\rvert$							$\frac{3}{10}\sqrt{\frac{3}{\pi}}$	$-\frac{1}{5}\sqrt{\frac{2}{\pi}}$		
2^{-+}	$\langle 2,0;0;2\rvert$									0	
2^{--}	$\langle 2,0;1;2\rvert$										$\frac{1}{\sqrt{8\pi}}$

Table B.2: The matrix $\mathcal{CT}$ is expressed in terms of the basis $|j, j_z; s; l\rangle$. Due to the conservation laws presented in chapter 2, it is possible to assume that $j_z = 0$. These values were computed with the help of *Mathematica*.

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