Master Thesis

Lattice study of possibly existing tetraquarks

Annabelle Uenver-Thiele

Institut für Theoretische Physik
Johann Wolfgang Goethe-Universität
Frankfurt am Main

September 2014

Supervisor: Prof. Dr. Marc Wagner
Second supervisor: Prof. Dr. Owe Philipsen

Erklärung

Nach § 30 (11) der Prüfungsordnung für den Bachelor- und Masterstudiengang Physik der Johann-Wolfgang-Goethe-Universität Frankfurt am Main erkläre ich, dass diese Arbeit von mir selbständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel verfasst wurde. Alle Stellen der Arbeit, die wörtlich oder sinngemäß aus Veröffentlichungen oder aus anderen fremden Texten entnommen wurden, sind als solche kenntlich gemacht. Weiter erkläre ich, dass die Arbeit nicht – auch nicht auszugsweise – für eine andere Prüfung verwendet worden ist.

Frankfurt am Main, den 10. September 2014

(Annabelle Uenver-Thiele)

Abstract

In this work we investigate possibly existing tetraquarks using gauge configurations with $N_f = 2 + 1 + 1$ Wilson twisted mass fermions at one lattice spacing. We also apply different smearing techniques and set all quark masses equal to the charm quark mass in order to save computation time.

First, we study the effective mass of a dynamic $\bar{c}c\bar{c}c$ 4-quark-state and do not find a clear signal of a bound state.
Based on studies of *static-staic-light-light* tetraquarks, we consider an *heavy-heavy-light-light* approximation of the static limit using light quarks and heavy anti-quarks. Computing the correlation functions of an *heavy-heavy-light-light* 4-quark operator, in analogy to the dynamic $\bar{c}c\bar{c}c$ 4-quark-state, leads to different masses for the isospin states $I = 0$ and $I = 1$. With this result we assume to have found attractive and repulsive channels for the two different states, which is suggested by the investigations of *static-static-light-light* tetraquarks.
Additionally, we construct alternative *heavy-heavy-light-light* operators that might be candidates for finding attractive channels by expressing *static-static-light-light* tetraquark operators in terms of *heavy-heavy-light-light* 4-quark-operators.
Computing the correlation functions for these constructed *heavy-heavy-light-light* operators we obtain slightly different masses, which might be a first indication of meson-meson-interactions.
However, we find that these constructed *heavy-heavy-light-light* 4-quark-operators do not represent the ground state but an overlap with excited states. Using different quark flavours and masses for the computations, i.e., a higher mass for the charm anti-quarks and a lower mass for the light quarks, the constructed *heavy-heavy-light-light* operators might excite the ground state, which is suggested by the numerical results and simple quantum mechanical considerations.

Kurzbeschreibung

Thema dieser Arbeit ist die Untersuchung möglicher Tetraquarkzustände basierend auf Eichkonfigurationen mit $N_f = 2 + 1 + 1$ Wilson twisted mass Fermionen für einen festen Gitterabstand. Wir verwenden außerdem verschiedene Smearing Techniken und setzen alle Quark Massen gleich der charm Quark Masse um Rechenzeit zu sparen.

Zunächst berechnen wir die effektive Masse für einen dynamischen $\bar{c}c\bar{c}c$ 4-Quark-Zustand und finden kein eindeutiges Signal für einen gebundenen Zustand.
Basierend auf Untersuchungen von *statisch-statisch-leichten-leichten* Tetraquarks, betrachten wir eine Annäherung an den statischen Limes unter Verwendung von leichten Quarks und schweren Anti-Quarks. Analog zu dem dynamischen $\bar{c}c\bar{c}c$ 4-Quark-Zustand, berechnen wir die Korrelationsfunktionen eines *schweren-schweren-leichten-leichten* Operators in der Annäherung an den statischen Limes, die zu unterschiedlichen Massen für die Isospin Zustände $I = 0$ und $I = 1$ führt. Aus diesem Ergebnis schließen wir, dass wir attraktive und repulsive Kanäle gefunden haben, basierend auf den Untersuchungen der *statisch-statisch-leichten-leichten* Tetraquarks.
Des Weiteren konstruieren wir alternative *schwere-schwere-leichte-leichte* Operatoren, die mögliche Kandidaten für Zustände mit attraktiven Kanälen repräsentieren, indem wir *statisch-statisch-leichte-leichte* Operatorn durch eine Linearkombination von *schweren-schweren-leichten-leichten* Operatorn ausdrücken. Bei Berechnung der Korrelationsfunktionen dieser *schweren-schweren-leichten-leichten* Operatorn erhalten wir kleine Massendifferenzen für die Isospin Zustände $I = 0$ und $I = 1$. Diese Beobachtung kann eine erste Indikation für eine Meson-Meson-Wechselwirkung darstellen.
Jedoch finden wir, dass diese konstruierten *schweren-schweren-leichten-leichten* Operatoren den Grundzustand nicht beschreiben, sondern viel mehr einen Überlapp mit angeregten Zuständen darstellen. Bei Verwendung anderer Quark Flavour und Massen für die Berechnungen, d.h. größere Massen für die schweren Anti-Quarks und geringere Massen für die leichten Quarks, erwarten wir, dass die berechneten *schweren-schweren-leichten-leichten* Operatoren den Grundzustand anregen, was durch die numerischen Ergebnisse und einfache quantenmechanische Überlegungen motiviert werden kann.

Contents

Chapter 1

Introduction and outline

An important field of Quantum Chromodynamics (QCD) is hadron spectroscopy. In this field investigations of the decays and masses of hadrons, such as mesons and baryons, are done.
In April 2014, the $Z(4430)$ hadron which consists of $c\bar{c}d\bar{u}$-quarks was found at the LHCb collaboration with a $13.9\,\sigma$ confidence level,[1]. However, there are several resonances that have been measured in experiments, which are not well understood yet, such as the nature of $X(3872), Y(3940), Y(4260)$ or $Y(4320)$ which seem to decay into charmonium states, i.e., $c\bar{c}$-mesons. With hadron spectroscopy physicists strive to understand these exotic hadrons like tetraquarks, molecules of hadrons or hybrids and glueballs in order to find answers to open questions of QCD.
With the PANDA experiment (anti Proton ANnihilation at DArmstadt) at the Gesellschaft für Schwerionenforschung (GSI), Germany, see [2], it will be possible to collect a large amount of charmonium states in order to explore these exotic states of hadrons.

An effective tool of investigating phenomena of QCD is Lattice QCD (LQCD). Using lattice computations it is possible to solve problems of QCD in the non-perturbative low-energy region. Since computations of ordinary meson spectra in LQCD have been very successful, tetraquarks and hybrids are also investigated with this technique.
On the one hand, the $\bar{c}c\bar{c}c$ tetraquark has been investigated using covariant Bethe–Salpeter equations in a coupled system. In [3] this tetraquark is predicted having a rather strong binding energy, but with an uncertainty of the same magnitude.
On the other hand, investigations of *static-static-light-light*-states, i.e., BB meson-pairs, have led to bound states, which are discussed in [4], [5]. Additionally, a coexisting master thesis investigates $B\bar{B}$ meson-pairs [6].

In this master thesis we discuss possibly existing tetraquarks consisting of four dynamical, twisted-mass fermions at a fixed lattice spacing.
First, we discuss 4-quark-states in general. Afterwards, we investigate the specific case of a 4-quark-state with four dynamical charm quarks and anti-quarks in order to compare our result with the result found in [3]. We also relate a *static-static-light-light* tetraquark

to an *heavy-heavy-light-light* 4-quark-state whose operator will be constructed in this work.
In order to find tetraquark candidates, we compare the effective masses of *heavy-heavy-light-light* 4-quark-states for different isospin quantum numbers $I = 0$ and $I = 1$. Since different masses imply to have either an attractive or repulsive potential, we are able to estimate which channels might have a bound state.

Chapter 2

Technical framework

2.1 Notation

In this work, we use upper indices a, b, c for denoting colour indices, lower indices A, B, C for spin indices, and upper indices $(i), (j), (k), (l)$ for flavour indices. Thus, the components of a fermion field are denoted by

$$\psi_A^{a,(m)}. \tag{2.1.1}$$

In order to describe operators we use the following notation:
Q denotes a static quark, q a quark with finite mass, l a light quark and more specifically we refer to an up, down or charm quark as u, d, or c respectively.

Furthermore, we refer to a *static-static-light-light* tetraquark as *static-light* tetraquark and to *heavy-heavy-light-light* 4-quark-state as *heavy-light* 4-quark-state.

2.2 Dirac matrices

Throughout this work, we use the chiral representation of the Dirac matrices for our calculations, i.e.,

$$\gamma_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix}$$

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2.3 Lattice setup

For our computations 100 different gauge link configurations of the ensemble $A40.24$ with the Iwasaki gauge action [7] generated by the European Twisted Mass Collaboration (ETMC) with $N_f = 2+1+1$ flavours of twisted mass quarks and the parameters given in table 2.1 have been used, cf. [8].

$T \times L^3$	β	κ	μ_l	μ_σ	μ_δ
48×24^3	1.9	0.16327	0.004	0.15	0.19

Table 2.1: Parameters of the gauge configurations.

The light degenerate quark doublet (u, d) is described by the standard Wilson twisted mass action [9]

$$S_{\text{light}}\left[\chi^{(l)}, \bar{\chi}^{(l)}, U\right] = a^4 \sum_x \bar{\chi}^{(l)}(x)\left(D_W(m_0) + i\mu\gamma_5\tau_3\right)\chi^{(l)}(x), \tag{2.3.1}$$

whereas the heavy sea quark doublet (c, s) is described by the twisted mass formulation for non-degenerated quarks [10]

$$S_{\text{heavy}}^{\text{sea}}\left[\chi^{(h)}, \bar{\chi}^{(h)}, U\right] = a^4 \sum_x \bar{\chi}^{(h)}(x)\left(D_W(m_0) + i\mu_\sigma\gamma_5\tau_1 + \tau_3\mu_\delta\right)\chi^{(h)}(x) \tag{2.3.2}$$

with D_W denoting the standard Wilson Dirac operator in both cases

$$D_W(m_0) = \frac{1}{2}\left(\gamma_\mu\left(\nabla_\mu + \nabla_\mu^*\right) - a\nabla_\mu^*\nabla_\mu\right) + m_0 \tag{2.3.3}$$

and $\chi^{(l)} = (\chi^{(u)}, \chi^{(d)})$, $\chi^{(h)} = (\chi^{(c)}, \chi^{(s)})$ denoting the fermion fields in the twisted mass basis.

As discussed in [11],[12], in the valence sector we will use a similar action for degenerate quarks as in eq. (2.3.1),

$$S_{\text{heavy}}^{\text{valence}}\left[\chi^{(c)}, \bar{\chi}^{(c)}, U\right] = a^4 \sum_x \bar{\chi}^{(c)}(x)\left(D_W(m_0) + i\mu\gamma_5\tau_3\right)\chi^{(c)}(x) \tag{2.3.4}$$

with degenerate twisted mass quark doublets $\chi^{(c)} = \left(\chi^{(c^+)}, \chi^{(c^-)}\right)$.

In order to save computation time, we use the mass of charm-quarks for all our computations with $\mu_c = 0.27678$.

2.4 Operators on the lattice

Since we use quarks in the twisted mass basis at maximal twist, we need to transform the physical basis into the twisted mass basis. For a more detailed discussion of the twisted mass formulation, see [4],[5],[11].

The transformation of $\psi^{(c)}$ from the physical basis into the fermion fields $\chi^{(c)}$ in the twisted mass basis is done in the following way:

$$\psi^{(c)} = \begin{pmatrix} \psi^{c^+} \\ \psi^{c^-} \end{pmatrix} \qquad \chi^{(c)} = \begin{pmatrix} \chi^{c^+} \\ \chi^{c^-} \end{pmatrix}$$

$$\psi \to e^{i\gamma_5 \tau_3 \frac{\omega}{2}} \chi \tag{2.4.1}$$

$$\bar{\psi} \to \bar{\chi} e^{i\gamma_5 \tau_3 \frac{\omega}{2}} \tag{2.4.2}$$

with ω denoting the twist angle, which has been tuned to maximal twist, i.e., $\omega = \frac{\pi}{2}$ and $\tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.
Transforming an operator from the physical into the twisted mass basis yields to

$$\bar{\psi}^{c^+} \gamma_5 \psi^{c^-} \to \bar{\chi}^{c^+} e^{i\gamma_5 \frac{\omega}{2}} \gamma_5 e^{-i\gamma_5 \frac{\omega}{2}} \chi^{c^-} \tag{2.4.3}$$

$$= \bar{\chi}^{c^+} \gamma_5 \chi^{c^-} \tag{2.4.4}$$

since the γ_5-matrices commute with each other.

In general we consider operators of the type

$$\bar{\psi}^{c^+} \Gamma_{\text{phy}} \, \psi^{c^-} = \bar{\chi}^{c^+} \Gamma_{\text{tm}} \, \chi^{c^-} \tag{2.4.5}$$

with the transformation of $\Gamma_{\text{phy}} \in \{\mathbb{1}, \gamma_0, \gamma_5, \gamma_0\gamma_5, \gamma_j, \gamma_0\gamma_j, \gamma_j\gamma_5\}$ given in table 2.2.

physical basis	twisted mass basis
γ_5	γ_5
$\gamma_0\gamma_5$	$-i\gamma_0$
$\mathbb{1}$	$\mathbb{1}$
γ_0	$-i\gamma_0\gamma_5$
γ_j	$-i\gamma_j\gamma_5$
$\gamma_0\gamma_j$	$\gamma_0\gamma_j$
$\gamma_j\gamma_5$	$-i\gamma_j\gamma_5$
$\gamma_0\gamma_j\gamma_5$	$\gamma_0\gamma_j\gamma_5$

Table 2.2: Transformation from the physical into the twisted mass basis.

In the following we use the notation of the physical basis for the operators.

Chapter 3

Theoretical backgrounds

3.1 Preliminaries

Let us consider a box of volume $T \times L^3$ in which we place two quarks and two anti-quarks. This **4-quark-state** can be characterised as either a

- **2-meson-state**, which implies, that there is an attractive or repulsive potential between the two mesons,
- **tetraquark state**, which indicates a bound state of four quarks.

In the following, we provide the mathematical framework for computing correlators of meson- and 4-quark-states on the lattice and interpreting the results. The effective mass can then be extracted from the correlation function.
First, we illustrate the different correlation functions for an η_c meson and a general 4-quark-state.
Then, we introduce and discuss different types of propagator diagrams, which correspond to the correlator of a 4-quark-state. These diagrams show the different possibilities of contracting two fermions.

Afterwards, we consider a 4-quark-state consisting of four dynamical quarks of finite mass with different flavour indices in general. As specific cases, we consider on the one hand two up and/or down quarks and two identical charm anti-quarks and on the other hand four dynamical charm quarks.
In the first case we can relate the isospin quantum numbers $I = 0$ and $I = 1$ to different compositions of diagram types.

3.2 Correlation functions

In order to calculate the effective mass of a meson or 4-quark-state we need to compute the correlation function of creation operators acting on the vacuum state.

$$\mathcal{C}(t_2 - t_1) = \langle \Omega | \, \mathcal{O}^\dagger(t_2) \mathcal{O}(t_1) \, | \Omega \rangle \tag{3.2.1}$$
$$= \sum_n \langle \Omega | \, \mathcal{O}^\dagger(t_2) \, | n \rangle \, \langle n | \, \mathcal{O}(t_1) \, | \Omega \rangle \, e^{-(E_n - E_\Omega)(t_2 - t_1)} \tag{3.2.2}$$

with $|\Omega\rangle$ denoting the vacuum state and E_Ω denoting the energy of the vacuum state. In the limit of $t \to \infty$ we can extract the mass of the ground state E_0, since higher order terms of E_n vanish.

$$m_{\text{eff}} = \ln \left(\frac{C(t)}{C(t+a)} \right) \tag{3.2.3}$$
$$m = \lim_{t \to \infty} m_{\text{eff}} \tag{3.2.4}$$

In the following, we first discuss the more simple case of calculating the correlation function of an η_c-meson. Afterwards, we calculate a general correlation function of a 4-quark-state.

3.2.1 η_c meson

An η_c meson consisting of a charm quark and anti-quark, is described by the following creation operator acting on the vacuum state:

$$\mathcal{O}^{\eta_c} = \sum_{\vec{x}} \bar{c}(\vec{x}) \, \gamma_5 \, c(\vec{x}) \tag{3.2.5}$$

with $\vec{x} = (x, y, z, t)$
In order to compute the effective mass, we need to consider the correlation function of eq. (3.2.1)

$$\mathcal{C}\,(t_2 - t_1) = \langle \Omega | \, \mathcal{O}^\dagger(t_2) \mathcal{O}(t_1) \, | \Omega \rangle \, .$$

with $\mathcal{O}(t_1) = \bar{\psi}^{(m)} \gamma_5 \, \psi^{(n)} \, (\vec{x}_1)$.
For the correlation function we obtain

$$\begin{aligned}
\mathcal{C}\,(t_2 - t_1) &= \sum_{\vec{x}_1, \vec{x}_2} \langle \Omega | \left(\bar{\psi}^{(m)} \gamma_5 \psi^{(n)} \right)^\dagger (\vec{x}_2) \left(\bar{\psi}^{(m)} \gamma_5 \psi^{(n)} \right) (\vec{x}_1) \, | \Omega \rangle \\
&= \sum_{\vec{x}_1, \vec{x}_2} \langle \Omega | \left(\bar{\psi}^{(n)} \gamma_0 \gamma_5^\dagger \gamma_0 \psi^{(m)} \right) (\vec{x}_2) \left(\bar{\psi}^{(m)} \gamma_5 \psi^{(n)} \right) (\vec{x}_1) \, | \Omega \rangle \\
&= - \left\langle \sum_{\vec{x}_1, \vec{x}_2} \text{Tr} \left\{ \gamma_0 \gamma_5^\dagger \gamma_0 \left(D^{(m)} \right)^{-1} (\vec{x}_2, \vec{x}_1) \, \gamma_5 \left(D^{(n)} \right)^{-1} (\vec{x}_1, \vec{x}_2) \right\} \right\rangle \\
&= - \left\langle \sum_{\vec{x}_1, \vec{x}_2} \text{Tr} \left\{ \gamma_5 \gamma_0 \gamma_5^\dagger \gamma_0 \left(D^{(m)} \right)^{-1} (\vec{x}_2, \vec{x}_1) \, \gamma_5 \gamma_5 \left(\left(D^{(n)} \right)^{-1} (\vec{x}_2, \vec{x}_1) \right)^\dagger \right\} \right\rangle .
\end{aligned} \tag{3.2.6}$$

By using the following relations

$$\begin{aligned}\mathrm{Tr}\left(DD^{\dagger}\right) &= \mathrm{Tr}\left(D^{\dagger}D\right)\\ \psi^{(n)}\left(\vec{x}_1\right)\bar{\psi}^{(n)}\left(\vec{x}_2\right) &\rightarrow \left(D^{(n)}\right)^{-1}\left(\vec{x}_1,\vec{x}_2\right) \text{ after integrating over all fermion fields}\\ \bar{\psi} &= \psi^{\dagger}\gamma_0\\ \gamma_5\gamma_0 &= -\gamma_0\gamma_5\end{aligned}$$

and taking into account that each permutation of fermions generates a change of sign we receive

$$\begin{aligned}\mathcal{C}\left(t_2-t_1\right) &= \left\langle \sum_{\vec{x}_1,\vec{x}_2} \mathrm{Tr}\left\{\left(\left(D^{(n)}\right)^{-1}\left(\vec{x}_2,\vec{x}_1\right)\right)^{\dagger}\left(D^{(m)}\right)^{-1}\left(\vec{x}_2,\vec{x}_1\right)\right\}\right\rangle\\ &= \left\langle \sum_{\vec{x}_1,\vec{x}_2}\left(\left(\left(D^{(n)}\right)^{ab}_{AB}\right)^{-1}\left(\vec{x}_2,\vec{x}_1\right)\right)^{\dagger}\left(\left(D^{(m)}\right)^{ab}_{AB}\right)^{-1}\left(\vec{x}_2,\vec{x}_1\right)\right)\right\rangle\\ &= \left\langle \sum_{\vec{x}_2}\phi^{*}\left[b,B,\vec{x}_1\right]^{a}_{A}\left(\vec{x}_2\right)\phi\left[b,B,\vec{x}_1\right]^{a}_{A}\left(\vec{x}_2\right)\right\rangle \qquad (3.2.7)\end{aligned}$$

where the definition of *point-like sources* is used, i.e., defining 12 point sources that are located at the same space-time-point $\vec{x}_1$

$$\xi\left[b,B,\vec{x}_1\right]^{c}_{C}\left(\vec{x}\right) = \delta^{bc}\delta_{BC}\delta\left(\vec{x}-\vec{x}_1\right). \qquad (3.2.8)$$

In order to compute the propagator, we need to solve the linear system

$$D^{ca}_{CA}\left(\vec{x},\vec{x}_2\right)\phi\left[b,B,\vec{x}_1\right]^{a}_{A}\left(\vec{x}_2\right) = \xi\left[b,B,\vec{x}_1\right]^{c}_{C}\left(\vec{x}\right). \qquad (3.2.9)$$

The propagator from x_1 to the space-time point x_2 is now given by

$$\left(D^{-1}\right)^{ab}_{AB}\left(\vec{x}_2,\vec{x}_1\right) = \phi\left[b,B,\vec{x}_1\right]^{a}_{A}\left(\vec{x}_2\right). \qquad (3.2.10)$$

Note, that we do not have to sum over x_1, since we can make use of translational invariance. For a more detailed discussion of *point sources*, see [13],[12]

With eq. (3.2.7) we can compute the correlator of an η_c meson. The result will be presented in ch. 4.1.

3.2.2 $\bar{q}q\bar{q}q$ 4-quark-state

Now, we calculate the correlation function of a general 4-quark-state. Therefore, we consider the following operator:

$$\mathcal{O}^{\bar{q}^{(m)}q^{(n)}\bar{q}^{(m)}q^{(n)}}_{\Gamma_1,\Gamma_2} = \sum_{\vec{x}}\left(\bar{q}^{(m)}_{A}(\vec{x})(\Gamma^1)_{AB}\, q^{(n)}_{B}(\vec{x})\right)\left(\bar{q}^{(m)}_{C}(\vec{x})(\Gamma^2)_{CD}\, q^{(n)}_{D}(\vec{x})\right) \qquad (3.2.11)$$

The 4-quark-state consists of four dynamical quarks which are all located at the same lattice site $\vec{x}$. The different quarks q and $\bar{q}$ are combined in spin-space by a combination of γ-matrices with 16 different possibilities of choosing $\Gamma_{1,2} \in \{\mathbb{1}, \gamma_0, \gamma_5, \gamma_0\gamma_5, \gamma_j, \gamma_0\gamma_j, \gamma_j\gamma_5\}$ which will be discussed in more detail in ch. 6.3.

In terms of the propagator, we need to compute

$$\begin{aligned}
\mathcal{C}\left(t_2-t_1\right) &= \left\langle \mathcal{O}^\dagger_{\Gamma^1,\Gamma^3}(t_2)\ \mathcal{O}_{\Gamma^2,\Gamma^4}(t_1)\right\rangle \\
\mathcal{C}\left(t_2-t_1\right) &= \sum_{\vec{x}_1,\vec{x}_2} \langle\Omega| \left((\bar{\psi}^{(m)})^a_A(\Gamma^1)_{AB}(\psi^{(n)})^a_B\right)^\dagger(\vec{x}_2)\left((\bar{\psi}^{(m)})^e_E(\Gamma^3)_{EF}(\psi^{(n)})^e_F\right)^\dagger(\vec{x}_2) \\
&\quad\cdot\left((\bar{\psi}^{(m)})^c_C(\Gamma^2)_{CD}(\psi^{(n)})^c_D\right)(\vec{x}_1)\left((\bar{\psi}^{(m)})^g_G(\Gamma^4)_{GH}(\psi^{(n)})^g_H\right)(\vec{x}_1)\,|\Omega\rangle \\
&= \sum_{\vec{x}_1,\vec{x}_2} \langle\Omega| \left((\bar{\psi}^{(n)})^a_B(\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}(\psi^{(m)})^a_A\right)(\vec{x}_2)\left((\bar{\psi}^{(n)})^e_F(\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}(\psi^{(m)})^e_E\right)(\vec{x}_2) \\
&\quad\cdot\left((\bar{\psi}^{(m)})^c_C(\Gamma^2)_{CD}(\psi^{(n)})^c_D\right)(\vec{x}_1)\left((\bar{\psi}^{(m)})^g_G(\Gamma^4)_{GH}(\psi^{(n)})^g_H\right)(\vec{x}_1)\,|\Omega\rangle
\end{aligned} \tag{3.2.12}$$

The correlation function of a 4-quark state can be represented by different propagator diagrams, which will be discussed in more detail in ch. 3.3.

$$\mathcal{C}\left(t_2-t_1\right) = \mathcal{C}^{\text{2meson}}(t_2-t_1)+\mathcal{C}^{\text{crossed}}(t_2-t_1)+\mathcal{C}^{\text{disconnected}}(t_2-t_1).$$

Now, we only consider the terms with the *two meson* and the *crossed* correlators.

$$\begin{aligned}
\mathcal{C}^{\text{2meson}} &= \Big\langle \sum_{\vec{x}_1,\vec{x}_2}(\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}(\Gamma^2)_{CD}\left((D^{(m)})^{ac}_{AC}\right)^{-1}(\vec{x}_2,\vec{x}_1)\left((D^{(n)})^{ca}_{DB}\right)^{-1}(\vec{x}_1,\vec{x}_2) \\
&\cdot(\gamma_0(\Gamma^3)\dagger\gamma_0)_{FE}(\Gamma^4)_{GH}\left((D^{(m)})^{eg}_{EG}\right)^{-1}(\vec{x}_2,\vec{x}_1)\left((D^{(n)})^{ge}_{HF}\right)^{-1}(\vec{x}_1,\vec{x}_2)\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_1,\vec{x}_2}(\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}(\Gamma^2)_{CD}\left((D^{(m)})^{ac}_{AC}\right)^{-1}(\vec{x}_2,\vec{x}_1)\,(\gamma_5)_{KB}\left(\left((D^{(n)})^{ac}_{KL}\right)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5)_{DL} \\
&\cdot(\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}(\Gamma^4)_{GH}\left((D^{(m)})^{eg}_{EG}\right)^{-1}(\vec{x}_2,\vec{x}_1)\,(\gamma_5)_{FM}\left(\left((D^{(n)})^{eg}_{MN}\right)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5)_{NH}\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_1,\vec{x}_2}(\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{KA}(\Gamma^2\gamma_5)_{CL}\left(\left((D^{(n)})^{ac}_{KL}\right)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger\left((D^{(m)})^{ac}_{AC}\right)^{-1}(\vec{x}_2,\vec{x}_1) \\
&\cdot(\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{ME}(\Gamma^4\gamma_5)_{GN}\left(\left((D^{(n)})^{eg}_{MN}\right)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger\left((D^{(m)})^{eg}_{EG}\right)^{-1}(\vec{x}_2,\vec{x}_1)\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_2}(\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{KA}(\Gamma^2\gamma_5)_{CL}\phi^*\left[c,L,x_1,t_1\right]^a_K(x_2,t_2)\,\phi\left[c,C,x_1,t_1\right]^a_A(x_2,t_2) \\
&\cdot(\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{ME}(\Gamma^4\gamma_5)_{GN}\phi^*\left[g,N,x_1,t_1\right]^e_M(x_2,t_2)\,\phi\left[g,G,x_1,t_1\right]^e_E(x_2,t_2)\Big\rangle.
\end{aligned} \tag{3.2.13}$$

Renaming L↔D, K↔B, N↔H, M↔F we find for the general two meson correlator of a 4-quark-state

$$\mathcal{C}^{\text{2meson}} = \Big\langle \sum_{\vec{x}_2} (\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}(\Gamma^2\gamma_5)_{CD}\phi^*\left[c,D,\vec{x}_1\right]^a_B(\vec{x}_2)\,\phi\left[c,C,\vec{x}_1\right]^a_A(\vec{x}_2) \cdot (\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}(\Gamma^4\gamma_5)_{GH}\phi^*\left[g,H,\vec{x}_1\right]^e_F(\vec{x}_2)\,\phi\left[g,G,\vec{x}_1\right]^e_E(\vec{x}_2)\Big\rangle. \tag{3.2.14}$$

In the case of the $\bar{c}c\bar{c}c$ 4-quark-state, we choose $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ to be γ_5.
Thus, we receive:

$$\mathcal{C}^{\text{2meson}}_{\bar{c}c\bar{c}c} = \Big\langle \sum_{\vec{x}_2} \Big\{\phi^*\left[c,C,\vec{x}_1\right]^a_A(\vec{x}_2)\,\phi\left[c,C,\vec{x}_1\right]^a_A(\vec{x}_2)\Big\}^2\Big\rangle. \tag{3.2.15}$$

For the crossed term we contract the two fermion fields in a different way:

$$\begin{aligned}
\mathcal{C}^{\text{crossed}} &= \Big\langle \sum_{\vec{x}_1,\vec{x}_2} \Big((D^{(n)})^{ga}_{HB}\Big)^{-1}(\vec{x}_1,\vec{x}_2)(\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}\Big((D^{(m)})^{ac}_{AC}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^2)_{CD} \\
&\cdot \Big((D^{(n)})^{ce}_{DF}\Big)^{-1}(\vec{x}_1,\vec{x}_2)(\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}\Big((D^{(m)})^{eg}_{EG}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^4)_{GH}\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_1,\vec{x}_2} (\gamma_5)_{HL}\left(\Big((D^{(n)})^{ag}_{KL}\Big)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5)_{BK}\,(\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}\Big((D^{(m)})^{ac}_{AC}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^2)_{CD} \\
&\cdot (\gamma_5)_{DN}\left(\Big((D^{(n)})^{ec}_{MN}\Big)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5)_{MF}\,(\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}\Big((D^{(m)})^{eg}_{EG}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^4)_{GH}\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_1,\vec{x}_2} \left(\Big((D^{(n)})^{ag}_{KL}\Big)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{KA}\Big((D^{(m)})^{ac}_{AC}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^2\gamma_5)_{CN} \\
&\cdot \left(\Big((D^{(n)})^{ec}_{MN}\Big)^{-1}(\vec{x}_2,\vec{x}_1)\right)^\dagger(\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{ME}\Big((D^{(m)})^{eg}_{EG}\Big)^{-1}(\vec{x}_2,\vec{x}_1)(\Gamma^4\gamma_5)_{GL}\Big\rangle \\
&= \Big\langle \sum_{\vec{x}_2} \phi^*\left[g,L,\vec{x}_1\right]^a_K(\vec{x}_2)\,(\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{KA}\phi\left[c,C,\vec{x}_1\right]^a_A(\vec{x}_2)\,(\Gamma^2\gamma_5)_{CN} \\
&\cdot \phi^*\left[c,N,\vec{x}_1\right]^e_M(\vec{x}_2)\,(\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{ME}\phi\left[g,G,\vec{x}_1\right]^e_E(\vec{x}_2)\,(\Gamma^4\gamma_5)_{GL}\Big\rangle
\end{aligned} \tag{3.2.16}$$

renaming L↔H, K↔B, N↔D, M↔F we obtain the general expression for the crossed correlator of a 4-quark-state

$$\mathcal{C}^{\text{crossed}} = \Big\langle \sum_{\vec{x}_2} \phi^*\left[g,H,\vec{x}_1\right]^a_B(\vec{x}_2)\,(\gamma_5\gamma_0(\Gamma^1)^\dagger\gamma_0)_{BA}\phi\left[c,C,\vec{x}_1\right]^a_A(\vec{x}_2)\,(\Gamma^2\gamma_5)_{CD} \cdot\ \phi^*\left[c,D,\vec{x}_1\right]^e_F(\vec{x}_2)\,(\gamma_5\gamma_0(\Gamma^3)^\dagger\gamma_0)_{FE}\phi\left[g,G,\vec{x}_1\right]^e_E(\vec{x}_2)\,(\Gamma^4\gamma_5)_{GH}\Big\rangle. \tag{3.2.17}$$

Again, in the case of the $\bar{c}c\bar{c}c$ 4-quark-state, we choose $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ to be γ_5.

Therefore we get:

$$\mathcal{C}^{\text{crossed}}_{\bar{c}c\bar{c}c} = \left\langle \sum_{\vec{x}_2} \Big(\phi^* \left[g, G, \vec{x}_1\right]^a_A (\vec{x}_2)\, \phi\left[c, C, \vec{x}_1\right]^a_A (\vec{x}_2)\, \phi^* \left[c, C, \vec{x}_1\right]^e_E (\vec{x}_2)\, \phi\left[g, G, \vec{x}_1\right]^e_E (\vec{x}_2) \Big) \right\rangle \tag{3.2.18}$$

The correlation function of a 4-quark-state will be discussed in ch. 4.2.

3.3 Diagrams and isospin

3.3.1 $\bar{q}q\bar{q}q$ 4-quark-state

Let us consider the operator of a 4-quark-state given in eq. (3.2.11)

$$\mathcal{O}^{\bar{q}^{(m)}q^{(n)}\bar{q}^{(m)}q^{(n)}}_{\Gamma_1,\Gamma_2} = \sum_{\vec{x}} \left(\bar{q}^{(m)}_A(\vec{x})(\Gamma^1)_{AB}\, q^{(n)}_B(\vec{x}) \right) \left(\bar{q}^{(m)}_C(\vec{x})(\Gamma^2)_{CD}\, q^{(n)}_D(\vec{x}) \right).$$

As previously indicated, the corresponding correlator can be represented by different types of propagator diagrams:

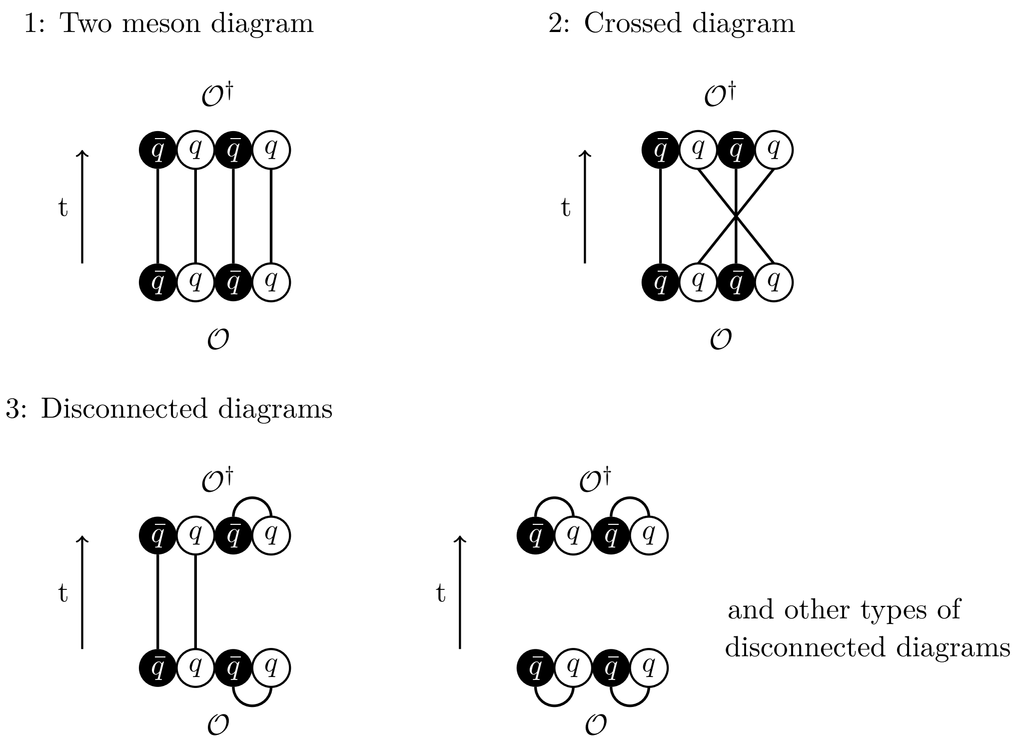
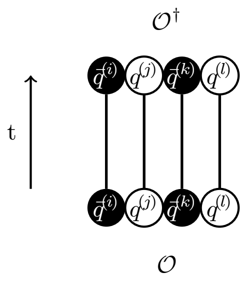


Figure 3.1: Possible diagrams describing a $\bar{q}q\bar{q}q$ 4-quark-state.

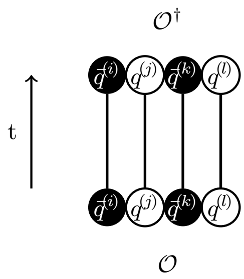
Depending on the chosen flavours of the quarks and anti-quarks, either the *two meson*, the *crossed*, the *disconnected* diagrams or a combination of those completely describe a certain correlator. In the following we discuss the different cases neglecting any types of *disconnected* diagrams.

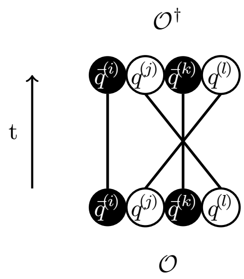
- Diagram 1 completely describes a correlator,
  if $\mathcal{C} = \langle \mathcal{O}_1^\dagger(t)\ \mathcal{O}_1(0)\rangle$ and
  if the quarks have different flavours **and** the anti-quarks have different flavours, i.e.,
  if $\bar{q}^{(i)} \neq \bar{q}^{(k)}$**and** $q^{(j)} \neq q^{(l)}$.



- Diagram 1 + 2 completely describe a correlator,
  if $\mathcal{C} = \langle \mathcal{O}_1^\dagger(t)\ \mathcal{O}_1(0)\rangle$ and
  if both quarks **or** both anti-quarks have the same flavours, i.e.
  if $\bar{q}^{(i)} = \bar{q}^{(k)}$ **or** $q^{(j)} = q^{(l)}$.

  Note, that we assume to have identical combinations of γ-matrices.

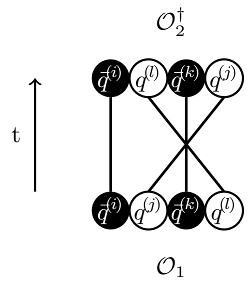




- Diagram 2 completely describes a correlator,
  if $\mathcal{C} = \langle \mathcal{O}_1^\dagger(t)\ \mathcal{O}_1(0)\rangle$ and
  if the quarks have different flavours **and** the anti-quarks have different flavours, i.e.,
  if $\bar{q}^{(i)} \neq q^{(j)}$ **and** $\bar{q}^{(k)} \neq q^{(l)}$. In this case we need two different operators in the correlation function
  $\mathcal{C}_{21}(t) = \langle \mathcal{O}_2^\dagger(t)\ \mathcal{O}_1(0)\rangle$:

$$\mathcal{O}_1^{(i),(j),(k),(l)} = \left(\bar{q}_A^{a,(i)} (\mathit{\Gamma}_1)_{AB}\ q_B^{a,(j)}\right) \left(\bar{q}_C^{c,(k)} (\mathit{\Gamma}_2)_{CD}\ q_D^{c,(l)}\right) \tag{3.3.1}$$

$$\mathcal{O}_2^{(i),(l),(k),(j)} = \left(\bar{q}_A^{a(i)} (\mathit{\Gamma}_3)_{AD}\ q_D^{a,(l)}\right) \left(\bar{q}_C^{c,(k)} (\mathit{\Gamma}_4)_{CB}\ q_B^{c,(j)}\right) \tag{3.3.2}$$



Note, that $\mathcal{O}_1$ and $\mathcal{O}_2$ still have the same quantum numbers, i.e. isospin I, angular momentum J and parity P.

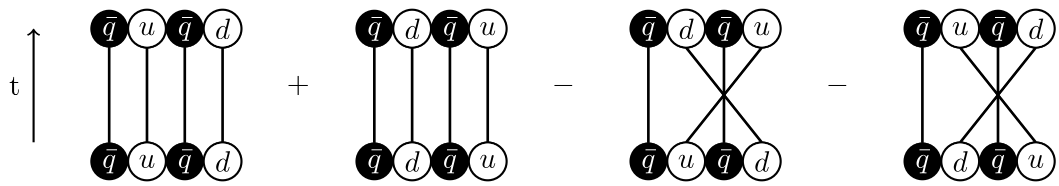
3.3.2 Isospin of 4-quark-states with two heavy and two light quarks

Let us now consider the case for a 4-quark-state in which the two quarks have either different or the same flavours, i.e., up and/or down, and the anti-quarks have the same.

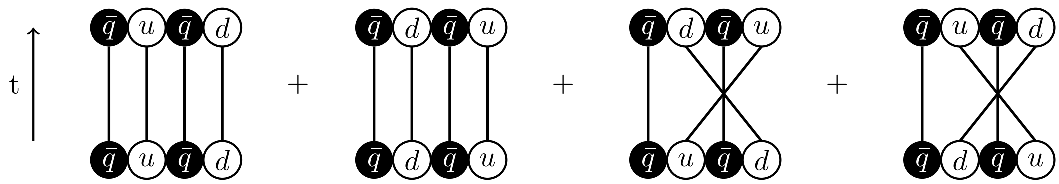
$$\mathcal{O}^{\bar{q} l^{(j)} \bar{q} l^{(k)}}_{\Gamma_1,\Gamma_2} = \sum_{\vec{x}} \left(\bar{q}_A(\vec{x})(\Gamma_1)_{AB}\, l^{(j)}_B(\vec{x})\right) \left(\bar{q}_C(\vec{x})(\Gamma_2)_{CD}\, l^{(k)}_D(\vec{x})\right) \tag{3.3.3}$$

This constellation of quarks and anti-quarks requires both types of diagrams, the *two meson* and *crossed* diagram. Since an up and down quark have a different isospin, the coupling of both leads to either a singlet or triplet state. For obtaining $I = 0$ we need $l^{(j)} l^{(k)} = ud - du$, whereas for $I = 1$ we can choose between $l^{(j)} l^{(k)} \in \{uu, dd, ud + du\}$. The different isospin quantum numbers result in different combinations of the *two meson* and *crossed* diagram, we can either add or subtract both diagrams.

- Diagram 1 - 2 → isospin I = 0, singlet state $(ud - du)$



- Diagram 1 + 2 → isospin I = 1, triplet state $(ud + du), uu, dd$



Thus, if we want to compute a 4-quark-state with the isospin quantum number $I = 0$, we need to subtract both diagrams and if we want to compute the isospin quantum

number $I = 1$, we need to add both ones.
By computing the masses of 4-quark-states with different isospin quantum numbers, we aim for not finding degenerated masses which is suggested by the studies of *static-light* tetraquarks [4],[14]. Different masses imply to have either an attractive or repulsive potential between the two mesons.

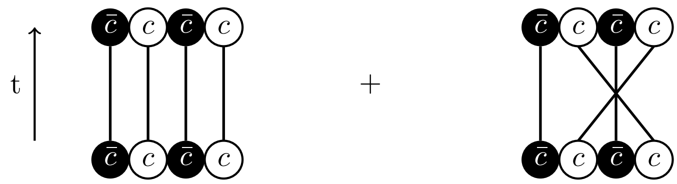
In ch. 5 we will discuss the result for the effective masses of these 4-quark-states.

3.3.3 $\bar{c}c\bar{c}c$ tetraquark candidate

Let us now discuss the specific case of a 4-quark-state consisting of four charm-quarks:

$$\mathcal{O}^{\bar{c}c\bar{c}c} = \sum_{\vec{x}} \Big(\bar{c}(\vec{x})\,\gamma_5\,c(\vec{x})\Big)\Big(\bar{c}(\vec{x})\,\gamma_5\,c(\vec{x})\Big) \tag{3.3.4}$$

Again, we neglect the *disconnected* diagrams. Therefore, this correlator can be completely described by combining the *two meson* and the *crossed* diagram:



In this case, there is only the combination of adding both diagram types, since we have charm quarks which do not have an isospin quantum number.
The result for the effective masses of the $\bar{c}c\bar{c}c$ tetraquark candidate will be discussed in ch. 4.2.

3.3.4 Expansion of diagrams in terms of energy eigenstates

In the next step we analyse how the different diagrams contribute to the effective mass of a 4-quark-state.

Let $\mathcal{C}_1$ be the correlator representing the *two meson* diagram and $\mathcal{C}_2$ the correlator representing the *crossed* diagram. Based on our definition of the isospin singlet and triplet states in ch. 3.3.2, the *two meson* diagram and the *crossed* diagram for themselves are a mixture of $I = 0$ and $I = 1$. Additionally, both correlators have an exponential decay and asymptotically the same mass, since they both are adequate correlators and the corresponding operators have the same quantum numbers.

$$\mathcal{C}_1(t) = A_1\, e^{-mt} + e^{-Mt} \overset{t\to\infty}{=} A_1\, e^{-mt} \tag{3.3.5}$$

$$\mathcal{C}_2(t) = A_2\, e^{-mt} + e^{-Mt} \overset{t\to\infty}{=} A_2\, e^{-mt}. \tag{3.3.6}$$

with m denoting the mass of the ground state and M the mass of higher order terms describing excited states, i.e. $m < M$.
It holds, that A_1 is positive definite and real, since the correlation function is of the type

$$\mathcal{C}_1(t) = \left\langle \mathcal{O}_1(t)^\dagger \mathcal{O}_1(t=0) \right\rangle > 0, \tag{3.3.7}$$

whereas A_2 can be any complex number, since

$$\mathcal{C}_2(t) = \left\langle \mathcal{O}_2(t)^\dagger \mathcal{O}_1(t=0) \right\rangle, \tag{3.3.8}$$

cf. (3.3.1),(3.3.2), does not necessarily need to be positive.
Constructing the combination of the two diagrams leads to

$$\left(\mathcal{C}_1 \pm \mathcal{C}_2\right)(t) = \left(A_1 \pm A_2\right) e^{-mt}. \tag{3.3.9}$$

From eq. (3.3.9) we can assume, that

(i) $A_2 \in \mathbb{R}$, because $A_1 + A_2 \in \mathbb{R}$
(ii) $|A_2| \leq A_1$, because $A_1 + A_2 \geq 0$,

since it holds, that

(i) $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 > 0$
(ii) $\mathcal{C}_1 \in \mathbb{R}, > 0$
(iii) $\mathcal{C} \in \mathbb{R} \Rightarrow \mathcal{C}_2 \in \mathbb{R}$.

Now, we want to find a condition for obtaining different masses for $\left(\mathcal{C}_1 \pm \mathcal{C}_2\right)(t)$ using eq. (3.3.5) and (3.3.6).

- For $A_2 = A_1$, we can obtain different masses for $I = 0$ and $I = 1$, since

$$\begin{aligned}\mathcal{C}_1 + \mathcal{C}_2 &= 2A_1\ e^{-mt}\\ \mathcal{C}_1 - \mathcal{C}_2 &= A_1\ e^{-Mt}.\end{aligned}$$

- In the case of $A_2 = -A_1$ we obtain

$$\begin{aligned}\mathcal{C}_1 + \mathcal{C}_2 &= A_1\ e^{-Mt}\\ \mathcal{C}_1 - \mathcal{C}_2 &= 2A_1\ e^{-mt}.\end{aligned}$$

- For $|A_2| < A_1$ there is a shift in the amplitude, which does not have any effect on the resulting mass.

If it holds, that $\mathcal{C}_2$ has the same mass as $\mathcal{C}_1$ in the leading term, i.e.,

$$\mathcal{C}_2(t) = \pm\left(A_1\, e^{-mt} - B_1\, e^{-Mt}\right) \quad \text{with } \; m \leq M \tag{3.3.10}$$

then, $\mathcal{C}_1 \pm C_2$ leads to different masses.

Based on eq. (3.3.10) it can hold that $\mathcal{C}_2 \approx 0$ and $\mathcal{C}_2 \ll \mathcal{C}_1$ if $m \approx M$ and $A_1 \approx B_1$. Then, by combining $\mathcal{C}_1$ and $\mathcal{C}_2$ the asymptotically dominating term can be eliminated. Thus, the correlator has to be of the type of eq. (3.3.10) in order to get different masses. To proof this assumption, we study the cases of $I = 0$ and $I = 1$ and plot the correlators of the different quantum numbers logarithmically in order to see how the slope of the two straight lines evolves in time. If there are different slopes, i.e., different masses, for large time separations, then eq. (3.3.10) is true.

3.4 Optimisation of smearing steps

When computing correlation functions in LQCD, different smearing techniques are used. With these techniques a better approximation of the ground state can be achieved for small time separations. For a more detailed discussion of smearing techniques, see [15]. In our computations, we use Gaussian and APE smearing. Since using these techniques results in higher computation times and higher errorbars, we need to optimise the number of smearing steps. An optimisation of the APE smearing was already done in earlier investigations and led to a sufficient number of smearing steps of $N_{\text{APE}} = 10$. Thus, we only need to optimise the number of smearing steps for Gaussian smearing.
In order to optimise the smearing steps, we consider the values of the effective masses for the time separation $t = 2$ for an increasing number of smearing steps.
In the following we consider the η_c meson and the $\bar{c}c\bar{c}c$ tetraquark candidate for optimising the number of smearing steps.

3.4.1 η_c meson

The effective masses for a different number of smearing steps for the η_c meson is given in the following figure:

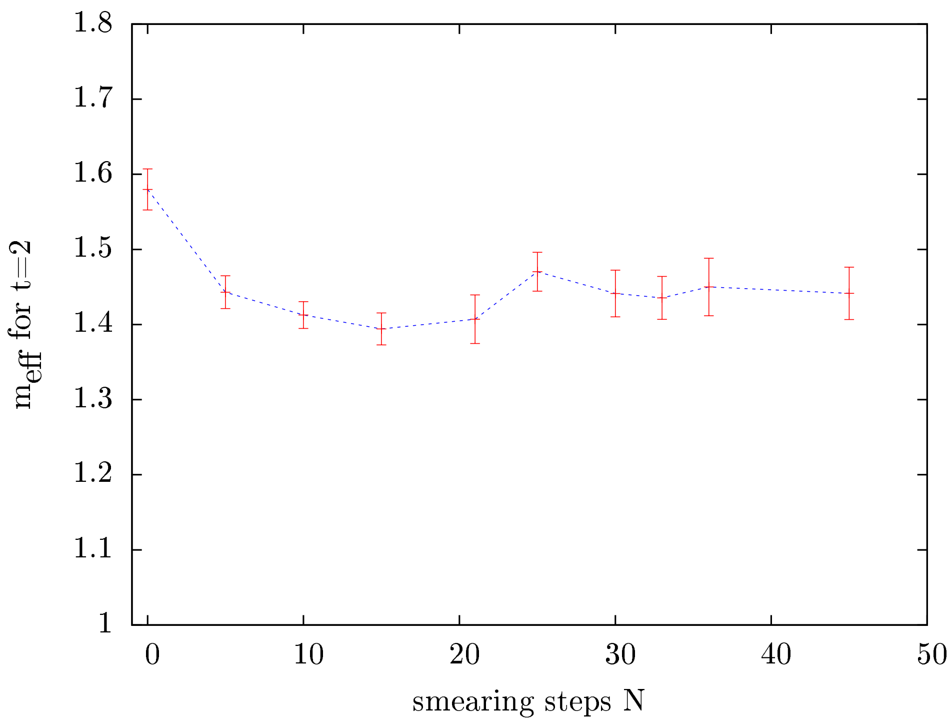


Figure 3.2: Optimisation of smearing steps using the values of the effective masses for the first time separations of the η_c meson.

In figure 3.2 we find, that the effective mass is nearly constant within the error for $N_{\text{Gauss}} > 5$. But we also see, that the error increases for a large number of smearing steps. Therefore, it seems optimal to choose $N_{\text{Gauss}} = 10 - 20$ smearing steps.

3.4.2 $\bar{c}c\bar{c}c$ 4-quark-state

For the effective masses of the 4-quark-state we obtain the following result:

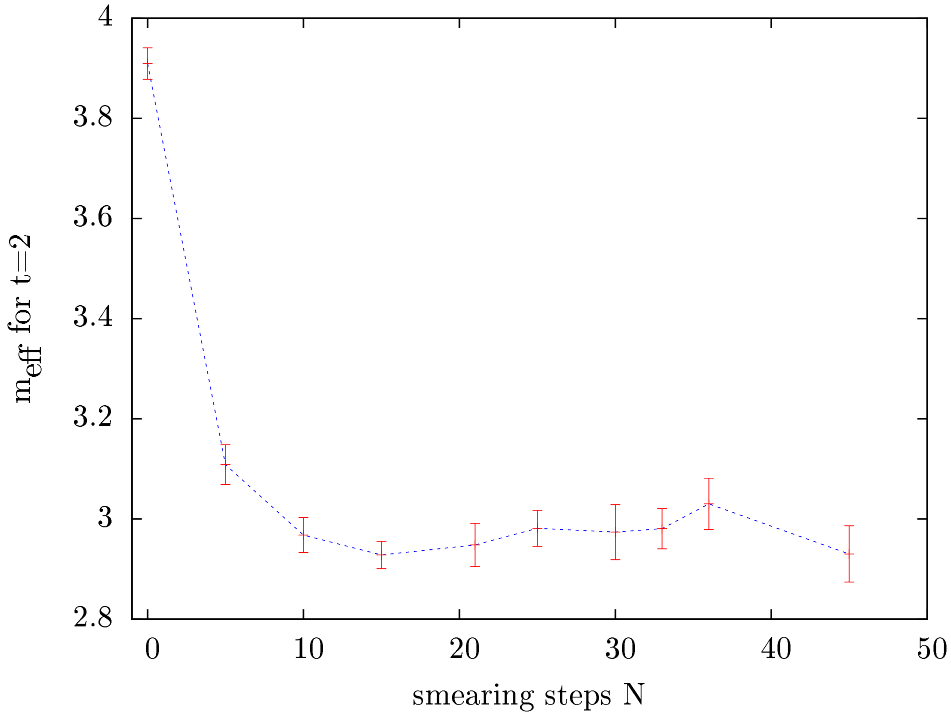


Figure 3.3: Optimisation of smearing steps using the values of the effective masses for the first time separations of the 4-quark-states.

We see in figure 3.3, that the effective mass for a 4-quark-state is also nearly constant for $N_{\text{Gauss}} > 10$. Since a larger number of smearing step results in higher errorbars and also in increasing computation times, we choose $N_{\text{Gauss}} = 15$ for our computations.

Chapter 4

Numerical results and analysis for the $\bar{c}c\bar{c}c$ tetraquark-candidate

4.1 Result for the η_c meson

Computing the effective mass of the correlator with the operator given in eq. (3.2.5) and the quantum numbers $I(J^P) = 0(0^-)$

$$\mathcal{O}^{\eta_c} = \sum_{\vec{x}} \bar{c}(\vec{x})\, \gamma_5\, c(\vec{x})$$

we obtain the following result:

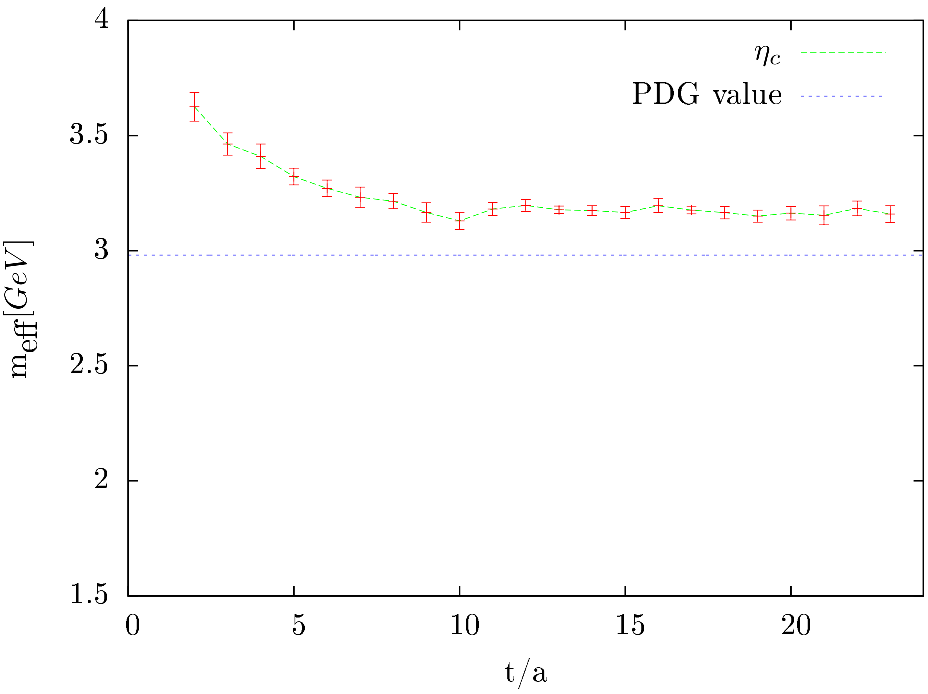


Figure 4.1: Effective mass of the η_c meson.

With our lattice computations of the correlation function using the operator given in eq. (3.2.7) using 10 configurations we find a mass of

$$m_{\eta_c} = (3.1730 \pm 0.012) \text{ GeV},$$

using a range of $t/a = 14 - 19$ for a constant fit.
A mass of

$$m_{\eta_c,PDG} = (2.981 \pm 0.001) \text{ GeV}$$

is given in the Physical Data Book [16].
For our computations, the charm-quark mass is not perfectly tuned to the physical mass of a charm-quark, since we are only interested in a qualitative investigation. Therefore, our result is about 6% larger than the result given in the PDG.

4.2 Result for the $\bar{c}c\bar{c}c$ tetraquark candidate

Computing the effective mass of the correlation function with the operator given in eq. (3.3.4)

$$\mathcal{O}^{\bar{c}c\bar{c}c} = \sum_{\vec{x}} \bar{c}(\vec{x})\, \gamma_5\, c(\vec{x})\; \bar{c}(\vec{x})\, \gamma_5\, c(\vec{x})$$

and the quantum numbers $I(J^P) = 0(0^+)$, yields to the following figure:

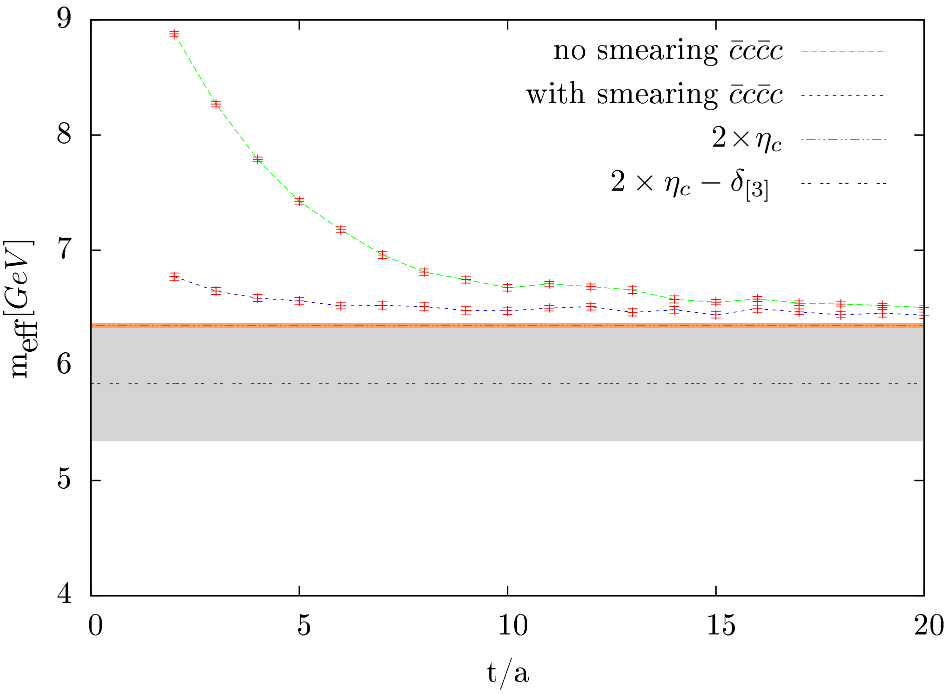


Figure 4.2: Comparison of the masses of $2 \times \eta_c$-meson and $\bar{c}c\bar{c}c$-tetraquark-candidate.

In figure 4.2 there are different masses shown. On the one hand the mass of two η_c

mesons with

$$m_{2\times\eta_c} \approx (6.3460 \pm 0.024) \text{ GeV}$$

and on the other hand the masses of a $\bar{c}c\bar{c}c$ state using smearing techniques.

$$\mathrm{m}_{\bar{c}c\bar{c}c}^{\mathrm{smea}} = (6.5088 \pm 0.0085)\,\mathrm{GeV}.$$

We see, that the masses of the 4-quark-states lie slightly above the mass of two η_c mesons.
The mass of a $\bar{c}c\bar{c}c$ state given in [3] is predicted to have a binding energy δ of

$$\delta_{[3]} = (0.7 \pm 0.5) \text{ GeV}.$$

Since in [3] the physical quark mass was used for the computations, a straight forward comparison is not sensible. To do so, we use the binding energy δ in order to obtain an estimation of a possible $\bar{c}c\bar{c}c$ state with the unphysical charm quark mass, i.e $m_{\bar{c}c\bar{c}c}^{\text{estimated}} = 2 \times m_{\eta_c} - \delta$.

The $\bar{c}c\bar{c}c$ 4-quark-state computed with the Bethe–Salpeter approach seems to form a bound state, because of the lower lying mass, in contrast to the results we obtain with our computations.
However, the effective mass of the correlator without using smearing techniques seems to still decrease for large values of t and seems to have not yet reached a plateau. This implies, that the ground state is not perfectly described. Considering the result we obtain by using smearing techniques, it becomes even more clear, that there is in fact an overlap between the trial state $\mathcal{O}\,|\Omega\rangle$ and excited states. This is indicated by the lower mass for small time separations using smearing techniques.
Note, that even by using smearing techniques, it is possible to still have an overlap with excited states, since we see that the mass of the $\bar{c}c\bar{c}c$ 4-quark-state is larger than $2 \times m_{\eta_c}$.

Chapter 5

Approximation of the static limit

5.1 Review: $\bar{Q}q\bar{Q}q$ tetraquark

Previous computations [14] showed, that there exists a tetraquark state consisting of two static anti-quarks and two light quarks which is described by the operator

$$\mathcal{O}^{\bar{Q}\,q^{(m)}\bar{Q}\,q^{(m)}} = (\mathcal{C}\Gamma^1)_{AB}(\Gamma^2)_{CD}\left(\bar{Q}_C(x_1)\,q_A^{(m)}(x_1)\right)\left(\bar{Q}_D(x_2)\,q_B^{(n)}(x_2)\right) \tag{5.1.1}$$

choosing $\Gamma^1 = \gamma_5 + \gamma_0\gamma_5$, $\Gamma^2 = \mathbb{1}$, the charge conjugation $\mathcal{C} = \gamma_0\gamma_2$ and $q^{(m)}, q^{(n)} \in \{u, d\}$.
The quantum numbers of the *static-light* tetraquark state are $I(J^P) = 0(0^+)$.
In contrast to the previous discussed case of four charm quarks that are located at the same lattice site, the two mesons are now located on different sites, because of the two static quarks.
Note, that in the static limit we have another spin structure than in the dynamical case due to different symmetries and quantum numbers. Since there is a degeneracy with respect to the static spin, we can only combine the two light quarks in spin space with each other.

5.2 The static limit of the $\bar{c}c\bar{c}c$ tetraquark candidate

Since we did not find a clear signal for a bound state of the dynamical $\bar{c}c\bar{c}c$ 4-quark-state, we consider now a theoretical approximation of the static limit. Since in the static limit there exists a bound state, we expect to be able to find attractive potentials by using an approximation of the static limit for the $\bar{c}c\bar{c}c$ 4-quark-state.
We assume to have two dynamic *light* quarks and two *heavy* anti-quarks, although we use the charm quark mass for all the computations in order to save CPU time.
Using this model, we can compare the two isospin states $I = 0$ and $I = 1$, as discussed in ch. 3.3, since we now have light quarks with isospin quantum numbers.
The operator we consider in the following is of the type

$$\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}} = \bar{c}_A(\gamma_5)_{AB}\,l_B^{(j)}\,\bar{c}_C(\gamma_5)_{CD}\,l_D^{(k)}. \tag{5.2.1}$$

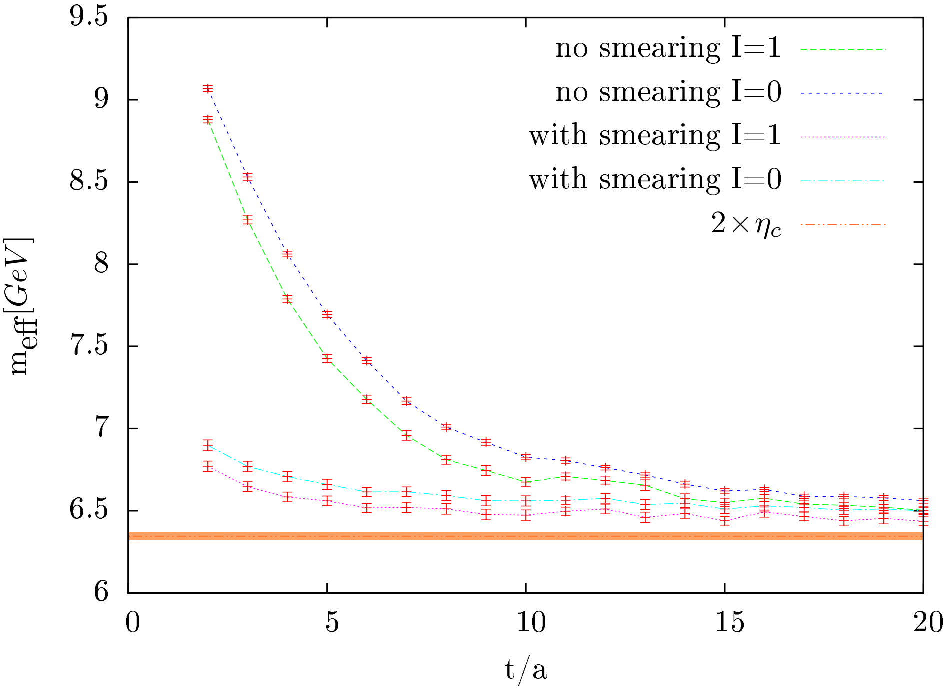


Figure 5.1: Comparison of the effective masses for $I = 0$ and $I = 1$ with the operator of eq. (5.2.1) with and without smearing.

The masses we obtain using APE and Gaussian smearing for the two different isospin states are indeed slightly different:

$$\mathrm{m}_{\mathrm{I=0}}^{\mathrm{smea}} = (6.5088 \pm 0.0085)\,\mathrm{GeV}$$
$$\mathrm{m}_{\mathrm{I=1}}^{\mathrm{smea}} = (6.4499 \pm 0.0105)\,\mathrm{GeV}.$$

Calculating the difference between the masses using the *Jackknife* method for the calculation of errors [17], we find

$$\mathrm{m}_{I=1}^{\mathrm{smea}} - \mathrm{m}_{I=0}^{\mathrm{smea}} = (0.0588 \pm 0.0041)\,\mathrm{GeV}.$$

In figure 5.1 it is again indicated, that the operator without using smearing techniques represents an overlap with excited states because of the same reason as mentioned in ch. 4.2.

However, the operator of the correlation function using smearing techniques does not perfectly excite the ground state, since for time separations in the range of $t/a = 15 - 20$ it is not clear, if a plateau is already reached.

For the difference of the masses using APE and Gaussian smearing we have a confidence level of $\approx 14\sigma$. Therefore, we can assume that the the mass of the $I = 0$ is indeed about 60 MeV smaller than the mass of the $I = 1$ state. We also observe, that the effective masses of the 4-quark-state using smearing techniques still lie above the mass of two η_c mesons. In ch. 9 a possible interpretation of this observation will be given.

As discussed in ch. 3.3.4, we now compare the slopes of theirs correlators.

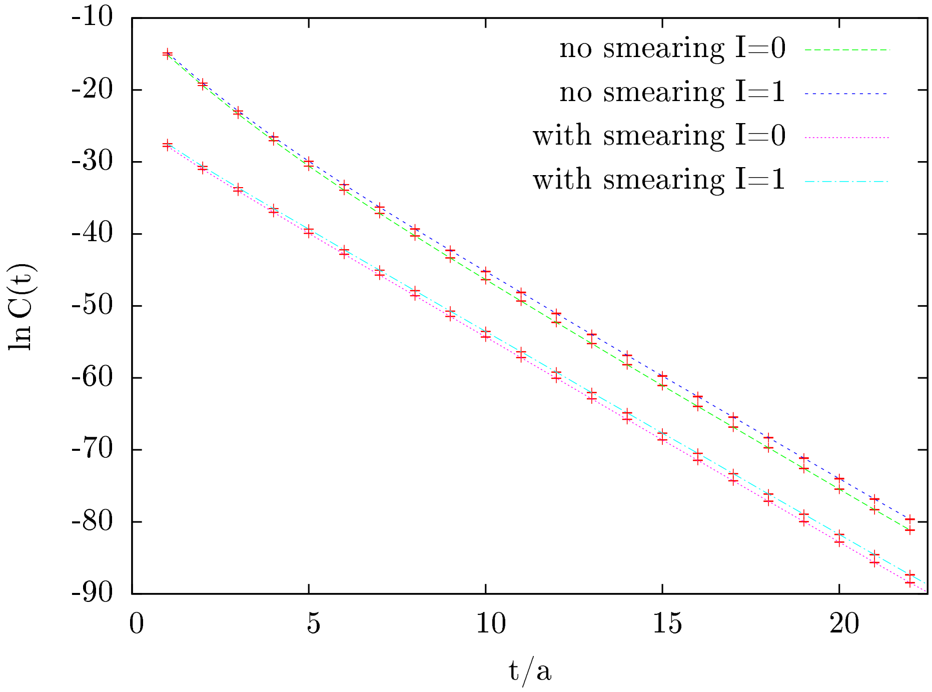


Figure 5.2: Logarithmic comparison of the correlator for $I = 0$ and $I = 1$ with the operator of eq. (5.2.1) with and without smearing.

Comparing the results of the logarithmic correlator for $I = 0$ and $I = 1$ with and without smearing, we see, that for small time separations the slopes, which correspond to the masses, are similar. However, for increasing values of t, the slopes of the straight lines differ due to higher order terms $\sim m - M$. This is in accordance with our conclusion in ch. 3.3.4.

Based on studies of the *static-light* tetraquark we know, that switching one quantum number by keeping all the others the same, like switching the isospin from $I = 0$ to $I = 1$, turns also an attractive potential into a repulsive and vice versa.

Thus, for the *heavy-light* operator we can have the following cases:

- The $I = 0$ state with the lower mass leads to an attractive and the $I = 1$ state with the higher mass to a repulsive potential [4],
- The $I = 0$ state with the lower mass is less repulsive than the $I = 1$ state with the higher mass,
- The $I = 0$ state with the lower mass is more attractive than the $I = 1$ state with the higher mass.

Since we did not find a signal for of a bound state for a $\bar{c}\, l^{(j)} \bar{c}\, l^{(k)}$ state, we construct an operator, that might be a candidate for finding a tetraquark state in the static limit.

Chapter 6

Representation of *static-light* operators in terms of *heavy-light* operators

6.1 Relation between *static-light* and *heavy-light* states

Since we know that there exists a bound state of tetraquarks consisting of two static and two light quarks, we try to express a *static-light* operator in terms of a *heavy-light* operator, i.e., a 4-quark-operator consisting of two heavy anti-quarks, such as charm quarks, and of two light quarks, such as up or down quarks:

$$\mathcal{O}^{\bar{Q}\,q^{(m)}\bar{Q}\,q^{(n)}} = \left(\mathcal{C}\Gamma^1\right)_{AB}\left(\Gamma^2\right)_{CD}\left(\bar{Q}_C\,q_A^{(m)}\right)\left(\bar{Q}_D\,q_B^{(n)}\right) \tag{6.1.1}$$

$$\approx \sum_{\lambda,\Gamma_1,\Gamma_2}\lambda\left(\bar{c}_A\left(\Gamma_1\right)_{AB}\,l_B^{(j)}\right)\left(\bar{c}_C\left(\Gamma_2\right)_{CD}\,l_D^{(k)}\right) \tag{6.1.2}$$

The relation expressed in eq. (6.1.2) is known as *Fierz identity.* With this identity we are able to express a product of bilinearforms of two spinors in terms of a linearcombination with products of bilinearforms of spinors. More details are given in [18].

Note that

- we omit the spacial arguments,
- we choose upper indices $1, 2$ of Γ in the *static-light* operator and lower indices $1, 2$ in the *heavy-light* operator,
- $q^{(m)}, q^{(n)}, l^{(j)}, l^{(k)} \in \{u, d\}$,

Thus, for the linear combination of the *heavy-light* 4-quark-state we consider the diagrams according to ch. 3.3.2.

6.2 Method of constructing the linear combination of the *heavy-light* operators

In order to construct the linear combination of the *heavy-light* operator in eq. (6.1.2) we consider a matrix multiplication of the type:

$$A\,\vec{x} = \vec{y}$$

where $\vec{x}, \vec{y} \in (1 \times N)$ and $A \in (N \times N)$.

Vector $\vec{x}$ consists of all spin-index combinations of the 4 quarks $\bar{Q}_i, q_j, \bar{Q}_k, q_l$ with $i, j, k, l = 1, 2, 3, 4$, whereas the components of $\vec{y}$ represent all possible *static-light* operators that describe a bound state.

Step 1

We can construct a matrix A, such that the components of the first row correspond to the coefficients of the terms in the *static-light* operator:

$$\underbrace{\begin{pmatrix} -1 & 0 & \cdots & +1 \\ 0 & -1 & \cdots & 0 \\ \vdots & \cdots & & \\ \vdots & & & \\ +1 & 0 & \cdots & -1 \end{pmatrix}}_{\substack{\text{Matrix A with coefficients} \\ \text{in the terms of} \\ \text{the first operator} \\ \mathcal{O}_{1,\text{static}} \text{ in the first row}}} \underbrace{\begin{pmatrix} \bar{Q}_1\, q_1\, \bar{Q}_1\, q_1 \\ \bar{Q}_1\, q_1\, \bar{Q}_1\, q_2 \\ \vdots \\ \vdots \\ \bar{Q}_4\, q_4\, \bar{Q}_4\, q_4 \end{pmatrix}}_{\substack{\vec{x}\text{: all combinations of spin} \\ \text{indices for the 4 quark flavours}}} = \underbrace{\begin{pmatrix} \mathcal{O}_{1,\text{static}} \\ \mathcal{O}_{2,\text{static}} \\ \vdots \\ \vdots \\ \mathcal{O}_{N,static} \end{pmatrix}}_{\substack{\vec{y}\text{: all possible operators} \\ \text{that form a bound state} \\ \text{or describe an attractive potential} \\ \text{in the static limit (eq.6.1.1)}}} \tag{6.2.1}$$

with $\mathcal{O}_{\text{static}} = \mathcal{O}^{\bar{Q}\,q^{(m)}\bar{Q}\,q^{(n)}} = (\mathcal{C}\Gamma^1)_{AB}(\Gamma^2)_{CD}\left(\bar{Q}_C\, q_A^{(m)}\right)\left(\bar{Q}_D\, q_B^{(n)}\right)$ and omitting the flavour indices of q in eq. (6.2.1).

Step 2

The next step is constructing a matrix B equivalently to matrix A, but with *heavy-light*-operators.

$$\underbrace{\begin{pmatrix} 0 & +1 & \cdots & -1 \\ +1 & 0 & \cdots & 0 \\ \vdots & \cdots & & \\ \vdots & & & \\ 0 & 0 & \cdots & -1 \end{pmatrix}}_{\substack{\text{Matrix B with coefficients} \\ \text{in the terms of} \\ \text{the first operator } \mathcal{O}_{1,\text{light}} \\ \text{in the first row}}} \begin{pmatrix} \bar{c}_1\, l_1\, \bar{c}_1\, l_1 \\ \bar{c}_1\, l_1\, \bar{c}_1\, l_2 \\ \vdots \\ \vdots \\ \bar{c}_4\, l_4\, \bar{c}_4\, l_4 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{O}_{1,\text{light}} \\ \mathcal{O}_{2,\text{light}} \\ \vdots \\ \vdots \\ \mathcal{O}_N \end{pmatrix}}_{\substack{\text{all possible heavy-light} \\ \text{4-quark-operators}}} \tag{6.2.2}$$

with $\mathcal{O}_{\text{light}} = \mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}} = \bar{c}_A(\Gamma_1)_{AB}\, l_B^{(j)}\, \bar{c}_C(\Gamma_2)_{CD}\, l_D^{(k)}$ and omitting the flavour indices of l in eq. (6.2.2).

Step 3

Now, we can combine the matrices A and B to find all the coefficients of λ_i of eq. (6.1.2).

$$B^{-1}A \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}}_{\mathcal{O}_{1,\text{static}}} = \underbrace{\begin{pmatrix} +1 \\ 0 \\ 0 \\ -1 \\ \vdots \end{pmatrix}}_{\substack{\text{coefficients } \lambda_j \\ \text{of eq. 6.1.2}}} \tag{6.2.3}$$

As a result of step 3, we have found a linear combination of possible *heavy-light* tetraquark operators that describe the *static-light* tetraquark operator of eq. (6.1.1). In the following we analyse this linear combination in order to find either an attractive or repulsive potential.

6.3 Physical Interpretation

Static-light-state

In the static limit, we have a degeneracy with respect to the static spin. Thus, for a given combination of Γ^1 we can have 16 possibilities of choosing Γ^2, which result in the potentials of the tetraquark states. However, there are eight combinations of γ-matrices that lead to a vanishing correlator. This can be seen by looking at the correlation function and inserting the static propagator:

$$\langle \Omega | \left(\mathcal{O}^{\bar{Q}q\bar{Q}q}(t)\right)^{\dagger} \ \mathcal{O}^{\bar{Q}q\bar{Q}q}(0) \, |\Omega\rangle \tag{6.3.1}$$

with

$$\mathcal{O}^{\bar{Q}\, q^{(m)} \bar{Q}\, q^{(n)}}(t) = \left(\mathcal{C}\Gamma^1(t)\right)_{AB} \left(\Gamma^2\right)_{CD} \ \bar{Q}^a_C(\vec{x}) \ q^{(m)a}_A(\vec{x}) \ \bar{Q}^b_D(\vec{y}) \ q^{(n)b}_B(\vec{y}) \tag{6.3.2}$$

$$(\mathcal{O}^{\bar{Q}\, q^{(n)} \bar{Q}\, q^{(m)}})^{\dagger}(t) = \left(\left(\gamma_0 \mathcal{C}\Gamma^1(t)\gamma_0\right)_{AB}\right)^{*} \left(\Gamma^2\right)_{CD} \ \bar{q}^{(n)b}_B(\vec{y}) \ Q^b_C(\vec{y}) \ \bar{q}^{(m)a}_A(\vec{x}) \ Q^a_D(\vec{x}). \tag{6.3.3}$$

Using the relation

$$Q_A(t,\vec{x}))\bar{Q}_B(0,\vec{x}) \sim \left(\frac{1+\gamma_0}{2}\right) \ U(t,\vec{x},0,\vec{x}) \tag{6.3.4}$$

we get an expression

$$\langle \Omega | \, (\mathcal{O}^{\bar{Q}q\bar{Q}q})^{\dagger} \, (t) \ \ \mathcal{O}^{\bar{Q}q\bar{Q}q} \, (0) \, |\Omega\rangle \sim \mathrm{Tr}\left(\Gamma^2 \frac{1+\gamma_0}{2} \ \Gamma^2 \frac{1+\gamma_0}{2}\right). \tag{6.3.5}$$

Based on eq. (6.3.5), we see that the combinations

$$\Gamma^2 \in \{\mathbb{1}, \ \gamma_0, \ \gamma_1\gamma_5, \ \gamma_2\gamma_5, \ \gamma_3\gamma_5, \ \gamma_0\gamma_1\gamma_5, \ \gamma_0\gamma_2\gamma_5, \ \gamma_0\gamma_3\gamma_5\}$$

lead to a non-vanishing correlator.

Heavy-light-state

By constructing linear combinations of γ-matrices for the static quarks $\bar{Q}$, we are able to eliminate half of the terms of the linear combination for the *heavy-light* operators in eq. (6.1.2).
The combinations we use are the following:

$$\Gamma^2 \in \{\gamma_0 \pm \mathbb{1}, \ \gamma_1\gamma_5 \pm \gamma_0\gamma_1\gamma_5, \ \gamma_2\gamma_5 \pm \gamma_0\gamma_2\gamma_5, \ \gamma_3\gamma_5 \pm \gamma_0\gamma_3\gamma_5\}$$

Considering the *heavy-light* operators, we can obtain the mesons* given in table 6.1.

*Remember, that we use charm-quarks for the computations and only assume in the theoretical model to have different quarks.

Meson	J^{PC}	Γ	mass
η_c	0^{-+}	$\gamma_5, \gamma_0\gamma_5$	2981.0 ± 1.1 MeV
J/Ψ	1^{--}	$\gamma_0\gamma_j, \gamma_j$	3096.916 ±0.01MeV
h_c	1^{+-}	$\gamma_0\gamma_5\gamma_j$	3525.41 ± 0.16 MeV
χ_{c0}	0^{++}	$\mathbb{1}$	3414.75 ± 0.31 MeV
χ_{c1}	1^{++}	$\gamma_5\gamma_j$	3510.66 ± 0.07 MeV
	0^{+-}	γ_0	experimentally not yet observed

Table 6.1: Quantum numbers of $\bar{c}c$ states.

Suggested by the investigations of the *static-light* tetraquarks, we choose $\Gamma^2 = \gamma_2\gamma_5 + \gamma_0\gamma_2\gamma_5 = (\gamma_5 + \gamma_0\gamma_5)\,\mathcal{C}$ and find:

$\Gamma^1 = \mathcal{C}(\gamma_5 + \gamma_0\gamma_5)$, $\Gamma^2 = (\gamma_5 + \gamma_0\gamma_5)\mathcal{C}$				
λ	Γ_1	Γ_2	meson	meson
+0.5	γ_5	γ_5	η_c	η_c
+0.5	γ_5	$\gamma_0\gamma_5$	η_c	η_c
+0.5	$\gamma_0\gamma_5$	γ_5	η_c	η_c
+0.5	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$	η_c	η_c
+0.5	γ_1	γ_1	J/ψ	J/ψ
+0.5	γ_1	$\gamma_0\gamma_1$	J/ψ	J/ψ
+0.5	γ_2	γ_2	J/ψ	J/ψ
+0.5	γ_2	$\gamma_0\gamma_2$	J/ψ	J/ψ
+0.5	γ_3	γ_3	J/ψ	J/ψ
+0.5	γ_3	$\gamma_0\gamma_3$	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_1$	γ_1	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_2$	γ_2	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_3$	γ_3	J/ψ	J/ψ
+0.5	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	J/ψ	J/ψ

Table 6.2: Terms of the linear combination describing the *static-light* operator of eq. (6.1.1).

In table 6.2 all different terms of the linear combination of eq. (6.1.2) are listed.
In the first four rows of table 6.2 we find the η_c mesons. Note, that J/Ψ mesons do also contribute to the linear combination, even if they are heavier than η_c mesons. Due to the fact, that the interaction between two mesons can be repulsive, the lightest state does not necessarily need to consist of the lightest mesons.

Result for the linear combination of the *heavy-light* operator

The 16 terms of table 6.2 represent the linear combination that describes the bound state of the *static-light* tetraquark operator of eq. (6.1.1), i.e.,

$$\mathcal{O}_1^{\bar{Q}\,q^{(m)}\bar{Q}\,q^{(n)}} = \bar{Q}\,(\gamma_5 + \gamma_0\gamma_5)\,\mathcal{C}\bar{Q}\;q^{(m)}\mathcal{C}\,(\gamma_5 + \gamma_0\gamma_5)\,q^{(n)} \tag{6.3.6}$$

$$\approx \mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}} \tag{6.3.7}$$

$$\begin{aligned} &= 0.5 \cdot (\bar{c}\,\gamma_5\,l^{(j)})\,(\bar{c}\,\gamma_5\,l^{(k)}) + 0.5 \cdot (\bar{c}\,\gamma_5\,l^{(j)})\,(\bar{c}\,\gamma_0\gamma_5\,l^{(k)}) \\ &\qquad + 0.5 \cdot (\bar{c}\,\gamma_0\gamma_5\,l^{(j)})\,(\bar{c}\,\gamma_5\,l^{(k)}) + ... \\ &= 0.5 \cdot \bar{c}\,(\gamma_5 + \gamma_0\gamma_5)\,l^{(j)}\;\bar{c}(\gamma_5 + \gamma_0\gamma_5)\,l^{(k)} \\ &\qquad + 0.5 \cdot \sum_{j=1}^{3} \bar{c}\,(\gamma_j + \gamma_0\gamma_j)\,l^{(j)}\;\bar{c}\,(\gamma_j + \gamma_0\gamma_j)\,l^{(k)} \end{aligned} \tag{6.3.8}$$

Depending on the chosen flavours of $q^{(m)}$ and $q^{(n)}$ we obtain a certain isospin for the *static-light* tetraquark:

$$q^{(m)}q^{(n)} = \begin{cases} I = 0, & \text{for } q^{(m)}q^{(n)} \in \{ud - du\} \\ I = 1, & \text{for } q^{(m)}q^{(n)} \in \{uu, dd, ud + du\} \end{cases} \tag{6.3.9}$$

Note, that this condition holds equivalently for $l^{(j)}$ and $l^{(k)}$.

As an example, we show the result of an alternative *static-light* operator choosing $\Gamma^1 = \mathcal{C}\,(\gamma_5 - \gamma_0\gamma_5)$ and $\Gamma^2 = (\gamma_5 + \gamma_0\gamma_5)\,\mathcal{C}$:

$\Gamma^1 = \mathcal{C}(\gamma_5 - \gamma_0\gamma_5)$ $\Gamma^2 = (\gamma_5 + \gamma_0\gamma_5)\mathcal{C}$				
λ	Γ_1	Γ_2	meson	meson
+0.5	$\mathbb{1}$	$\mathbb{1}$	χ_{c_0}	χ_{c_0}
+0.5	$\mathbb{1}$	γ_0	χ_{c_0}	
+0.5	γ_0	$\mathbb{1}$		χ_{c_0}
+0.5	γ_0	γ_0		
+0.5	$\gamma_1\gamma_5$	$\gamma_1\gamma_5$	χ_{c_1}	χ_{c_1}
+0.5	$\gamma_1\gamma_5$	$\gamma_0\gamma_1\gamma_5$	χ_{c_1}	h_c
+0.5	$\gamma_2\gamma_5$	$\gamma_2\gamma_5$	χ_{c_1}	χ_{c_1}
+0.5	$\gamma_2\gamma_5$	$\gamma_0\gamma_2\gamma_5$	χ_{c_1}	h_c
+0.5	$\gamma_3\gamma_5$	$\gamma_3\gamma_5$	χ_{c_1}	χ_{c_1}
+0.5	$\gamma_3\gamma_5$	$\gamma_0\gamma_3\gamma_5$	χ_{c_1}	h_c
+0.5	$\gamma_0\gamma_1\gamma_5$	$\gamma_1\gamma_5$	h_c	χ_{c_1}
+0.5	$\gamma_0\gamma_1\gamma_5$	$\gamma_0\gamma_1\gamma_5$	h_c	h_c
+0.5	$\gamma_0\gamma_2\gamma_5$	$\gamma_2\gamma_5$	h_c	χ_{c_1}
+0.5	$\gamma_0\gamma_2\gamma_5$	$\gamma_0\gamma_2\gamma_5$	h_c	h_c
+0.5	$\gamma_0\gamma_3\gamma_5$	$\gamma_3\gamma_5$	h_c	χ_{c_1}
+0.5	$\gamma_0\gamma_3\gamma_5$	$\gamma_0\gamma_3\gamma_5$	h_c	h_c

Table 6.3: Example of choosing an alternative *static-light* operator with $\Gamma^1 = \mathcal{C}(\gamma_5 - \gamma_0\gamma_5)$ and $\Gamma^2 = (\gamma_5 + \gamma_0\gamma_5)\mathcal{C}$, which leads to another linear combination of *heavy-light* operators.

In table 6.3 we find, that there are no η_c mesons in the linear combination of the *heavy-light* operator for the alternative choice of the *static-light* operator.

Chapter 7

Representation of a *heavy-light* operator in terms of *static-light* operators

In order to get a better understanding of the forces between two η_c mesons, let us now express this *heavy-light* operator in terms of *static-light* operators.
Thus, we are now interested in finding a *static-light* operator, that eliminates all terms with γ-combinations describing the J/ψ meson.

In addition to the previously discussed *static-light* operator with $\Gamma^1 = \mathcal{C}(\gamma_5 + \gamma_0\gamma_5)$ and $\Gamma^2 = (\gamma_5 + \gamma_0\gamma_5)\mathcal{C}$, we take into account three other similar operators. The results are listed in the following table:

Combination 1												
	$\mathcal{O}^{\gamma_1+\gamma_0\gamma_1}$			$\mathcal{O}^{\gamma_2+\gamma_0\gamma_2}$			$\mathcal{O}^{\gamma_3+\gamma_0\gamma_3}$			$\mathcal{O}^{\gamma_5+\gamma_0\gamma_5}$		
Γ^1	$\mathcal{C}(\gamma_1+\gamma_0\gamma_1)$			$\mathcal{C}(\gamma_2+\gamma_0\gamma_2)$			$\mathcal{C}(\gamma_3+\gamma_0\gamma_3)$			$\mathcal{C}(\gamma_5+\gamma_0\gamma_5)$		
Γ^2	$(\gamma_1+\gamma_0\gamma_1)\mathcal{C}$			$(\gamma_2+\gamma_0\gamma_2)\mathcal{C}$			$(\gamma_3+\gamma_0\gamma_3)\mathcal{C}$			$(\gamma_5+\gamma_0\gamma_5)\mathcal{C}$		
	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2
	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5
	$+0.5$	γ_5	$\gamma_0\gamma_5$	$+0.5$	γ_5	$\gamma_0\gamma_5$	$+0.5$	γ_5	$\gamma_0\gamma_5$	$+0.5$	γ_5	$\gamma_0\gamma_5$
	$+0.5$	$\gamma_0\gamma_5$	γ_5	$+0.5$	$\gamma_0\gamma_5$	γ_5	$+0.5$	$\gamma_0\gamma_5$	γ_5	$+0.5$	$\gamma_0\gamma_5$	γ_5
	$+0.5$	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$	$+0.5$	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$	$+0.5$	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$	$+0.5$	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$
	$+0.5$	γ_1	γ_1	-0.5	γ_1	γ_1	-0.5	γ_1	γ_1	$+0.5$	γ_1	γ_1
	$+0.5$	γ_1	$\gamma_0\gamma_1$	-0.5	γ_1	$\gamma_0\gamma_1$	-0.5	γ_1	$\gamma_0\gamma_1$	$+0.5$	γ_1	$\gamma_0\gamma_1$
	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2
	-0.5	γ_2	$\gamma_0\gamma_2$	$+0.5$	γ_2	$\gamma_0\gamma_2$	-0.5	γ_2	$\gamma_0\gamma_2$	$+0.5$	γ_2	$\gamma_0\gamma_2$
	-0.5	γ_3	γ_3	-0.5	γ_3	γ_3	$+0.5$	γ_3	γ_3	$+0.5$	γ_3	γ_3
	-0.5	γ_3	$\gamma_0\gamma_3$	-0.5	γ_3	$\gamma_0\gamma_3$	$+0.5$	γ_3	$\gamma_0\gamma_3$	$+0.5$	γ_3	$\gamma_0\gamma_3$
	$+0.5$	$\gamma_0\gamma_1$	γ_1	-0.5	$\gamma_0\gamma_1$	γ_1	-0.5	$\gamma_0\gamma_1$	γ_1	$+0.5$	$\gamma_0\gamma_1$	γ_1
	$+0.5$	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	-0.5	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	-0.5	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$+0.5$	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$
	-0.5	$\gamma_0\gamma_2$	γ_2	$+0.5$	$\gamma_0\gamma_2$	γ_2	-0.5	$\gamma_0\gamma_2$	γ_2	$+0.5$	$\gamma_0\gamma_2$	γ_2
	-0.5	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$+0.5$	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	-0.5	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$+0.5$	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$
	-0.5	$\gamma_0\gamma_3$	γ_3	-0.5	$\gamma_0\gamma_3$	γ_3	$+0.5$	$\gamma_0\gamma_3$	γ_3	$+0.5$	$\gamma_0\gamma_3$	γ_3
	-0.5	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	-0.5	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$+0.5$	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$+0.5$	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$

Table 7.1: Different *static-light* operators in terms of *heavy-light* operators.

Using the results of table 7.1, we can construct an operator equivalently to eq. (6.3.8):

$$\mathcal{O}_2^{\bar{Q}\,q^{(m)}\bar{Q}\,q^{(n)}} = \sum_{j=1}^{3} \bar{Q}\left(\gamma_j+\gamma_0\gamma_j\right)\mathcal{C}\bar{Q}\;q^{(m)}\mathcal{C}\left(\gamma_j+\gamma_0\gamma_j\right)q^{(n)} \tag{7.0.1}$$

$$\begin{aligned} &\approx \mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}} \\ &= 3\cdot\bar{c}\left(\gamma_5+\gamma_0\gamma_5\right)l^{(j)}\;\bar{c}\left(\gamma_5+\gamma_0\gamma_5\right)l^{(k)} \\ &\qquad -\sum_{j=1}^{3}\bar{c}\left(\gamma_j+\gamma_0\gamma_j\right)l^{(j)}\;\bar{c}\left(\gamma_j+\gamma_0\gamma_j\right)l^{(k)} \end{aligned} \tag{7.0.2}$$

Note that, the *static-light* operators with $\gamma_j+\gamma_0\gamma_j$ matrices do not describe a bound state, but they can have an attractive potential depending on the chosen isospin quantum number, cf. table 7.10 and 7.11.
Again, we need to distinguish between the flavours $q^{(m)}, q^{(n)}, l^{(j)}$ and $l^{(k)}$ and get the same condition as in eq. (6.3.9).

Now, with a linear combination of the four *static-light* operators we obtain *heavy-light* operators consisting of η_c mesons.

$$\frac{1}{2}\cdot\left(\mathcal{O}^{\gamma_1+\gamma_0\gamma_1}+\mathcal{O}^{\gamma_2+\gamma_0\gamma_2}+\mathcal{O}^{\gamma_3+\gamma_0\gamma_3}+\mathcal{O}^{\gamma_5+\gamma_0\gamma_5}\right). \tag{7.0.3}$$

Combination 1				
$\frac{1}{2}(\mathcal{O}^{\gamma_1+\gamma_0\gamma_1}+\mathcal{O}^{\gamma_2+\gamma_0\gamma_2}+\mathcal{O}^{\gamma_3+\gamma_0\gamma_3}+\mathcal{O}^{\gamma_5+\gamma_0\gamma_5})$				
λ	Γ_1	Γ_2	meson	meson
1	γ_5	γ_5	η_c	η_c
1	γ_5	$\gamma_0\gamma_5$	η_c	η_c
1	$\gamma_0\gamma_5$	γ_5	η_c	η_c
1	$\gamma_0\gamma_5$	$\gamma_0\gamma_5$	η_c	η_c

Table 7.2: Linear combination of four *static-light* operators that describe *heavy-light* operators with only η_c mesons.

Since we were using only 8 out of 16 possible combinations of γ-matrices for the static quarks in the *static-light* operator, terms with $\gamma_0\gamma_5$ in table 7.2 are not being eliminated. For the construction of the $\bar{c}c\bar{c}c$ operator of eq. (3.3.4) as a linear combination of *static-light* operators equivalent to the first row of table 7.2, we need to consider 12 other *static-light* operators. In the following we ignore the fact that two of the four Dirac components of the static quarks vanish, i.e.,

$$\bar{Q} \mathrel{\widehat{=}} \bar{Q}\frac{1+\gamma_0}{2} = \bar{Q}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \tag{7.0.4}$$

This means, that the relation between a *static-light* and *heavy-light* operator is actually a more general relation, which is also valid for heavy quarks Q with finite mass. Therefore, we also consider *static-light* operators, that actually lead to a vanishing correlator.

Combination 2												
	$\mathcal{O}^{\gamma_1-\gamma_0\gamma_1}$			$\mathcal{O}^{\gamma_2-\gamma_0\gamma_2}$			$\mathcal{O}^{\gamma_3-\gamma_0\gamma_3}$			$\mathcal{O}^{\gamma_5-\gamma_0\gamma_5}$		
Γ^1	$\mathcal{C}(\gamma_1-\gamma_0\gamma_1)$			$\mathcal{C}(\gamma_2-\gamma_0\gamma_2)$			$\mathcal{C}(\gamma_3-\gamma_0\gamma_3)$			$\mathcal{C}(\gamma_5-\gamma_0\gamma_5)$		
Γ^2	$(\gamma_1-\gamma_0\gamma_1)\mathcal{C}$			$(\gamma_2-\gamma_0\gamma_2)\mathcal{C}$			$(\gamma_3-\gamma_0\gamma_3)\mathcal{C}$			$(\gamma_5-\gamma_0\gamma_5)\mathcal{C}$		
	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2
	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5
	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$
	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5
	$+0.5$	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	$+0.5$	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	$+0.5$	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	$+0.5$	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$
	$+0.5$	γ_1	γ_1	-0.5	γ_1	γ_1	-0.5	γ_1	γ_1	$+0.5$	γ_1	γ_1
	-0.5	γ_1	$\gamma_0\ \gamma_1$	$+0.5$	γ_1	$\gamma_0\ \gamma_1$	$+0.5$	γ_1	$\gamma_0\ \gamma_1$	-0.5	γ_1	$\gamma_0\ \gamma_1$
	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2
	$+0.5$	γ_2	$\gamma_0\ \gamma_2$	-0.5	γ_2	$\gamma_0\ \gamma_2$	$+0.5$	γ_2	$\gamma_0\ \gamma_2$	-0.5	γ_2	$\gamma_0\ \gamma_2$
	-0.5	γ_3	γ_3	-0.5	γ_3	γ_3	$+0.5$	γ_3	γ_3	$+0.5$	γ_3	γ_3
	$+0.5$	γ_3	$\gamma_0\ \gamma_3$	$+0.5$	γ_3	$\gamma_0\ \gamma_3$	-0.5	γ_3	$\gamma_0\ \gamma_3$	-0.5	γ_3	$\gamma_0\ \gamma_3$
	-0.5	$\gamma_0\ \gamma_1$	γ_1	$+0.5$	$\gamma_0\ \gamma_1$	γ_1	$+0.5$	$\gamma_0\ \gamma_1$	γ_1	-0.5	$\gamma_0\ \gamma_1$	γ_1
	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$
	$+0.5$	$\gamma_0\ \gamma_2$	γ_2	-0.5	$\gamma_0\ \gamma_2$	γ_2	$+0.5$	$\gamma_0\ \gamma_2$	γ_2	-0.5	$\gamma_0\ \gamma_2$	γ_2
	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$
	$+0.5$	$\gamma_0\ \gamma_3$	γ_3	$+0.5$	$\gamma_0\ \gamma_3$	γ_3	-0.5	$\gamma_0\ \gamma_3$	γ_3	-0.5	$\gamma_0\ \gamma_3$	γ_3
	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$

Table 7.3: Different *static-light* operators in terms of *heavy-light* operators.

Combination 2		
$\frac{1}{2}(\mathcal{O}^{\gamma_1-\gamma_0\gamma_1}+\mathcal{O}^{\gamma_2-\gamma_0\gamma_2}$ $+\mathcal{O}^{\gamma_3-\gamma_0\gamma_3}+\mathcal{O}^{\gamma_5-\gamma_0\gamma_5})$		
λ	Γ_1	Γ_2
$+1$	γ_5	γ_5
-1	γ_5	$\gamma_0\ \gamma_5$
-1	$\gamma_0\ \gamma_5$	γ_5
$+1$	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$

Table 7.4: Linear combination of four *static-light* operators that describe *heavy-light* operators with only η_c mesons.

Combination 3												
	$\mathcal{O}^{(\gamma_1+\gamma_0\gamma_1)\gamma_5}$			$\mathcal{O}^{(\gamma_2+\gamma_0\gamma_2)\gamma_5}$			$\mathcal{O}^{(-\gamma_3-\gamma_0\gamma_3)\gamma_5}$			$\mathcal{O}^{(\mathbb{1}+\gamma_0)}$		
Γ^1	$C(\gamma_1+\gamma_0\gamma_1)\gamma_5$			$C(\gamma_2+\gamma_0\gamma_2)\gamma_5$			$C(-\gamma_3-\gamma_0\gamma_3)\gamma_5$			$C(\mathbb{1}+\gamma_0)$		
Γ^2	$-\gamma_5(\gamma_1-\gamma_0\gamma_1)\mathcal{C}$			$-\gamma_5(\gamma_2-\gamma_0\gamma_2)\mathcal{C}$			$\gamma_5(\gamma_3-\gamma_0\gamma_3)\mathcal{C}$			$(\mathbb{1}+\gamma_0)\mathcal{C}$		
	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2
	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5
	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$	-0.5	γ_5	$\gamma_0\ \gamma_5$
	$+0.5$	$\gamma_0\ \gamma_5$	γ_5	$+0.5$	$\gamma_0\ \gamma_5$	γ_5	$+0.5$	$\gamma_0\ \gamma_5$	γ_5	$+0.5$	$\gamma_0\ \gamma_5$	γ_5
	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$
	-0.5	γ_1	γ_1	$+0.5$	γ_1	γ_1	$+0.5$	γ_1	γ_1	-0.5	γ_1	γ_1
	$+0.5$	γ_1	$\gamma_0\ \gamma_1$	-0.5	γ_1	$\gamma_0\ \gamma_1$	-0.5	γ_1	$\gamma_0\ \gamma_1$	$+0.5$	γ_1	$\gamma_0\ \gamma_1$
	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2
	-0.5	γ_2	$\gamma_0\ \gamma_2$	$+0.5$	γ_2	$\gamma_0\ \gamma_2$	-0.5	γ_2	$\gamma_0\ \gamma_2$	$+0.5$	γ_2	$\gamma_0\ \gamma_2$
	$+0.5$	γ_3	γ_3	$+0.5$	γ_3	γ_3	-0.5	γ_3	γ_3	-0.5	γ_3	γ_3
	-0.5	γ_3	$\gamma_0\ \gamma_3$	-0.5	γ_3	$\gamma_0\ \gamma_3$	$+0.5$	γ_3	$\gamma_0\ \gamma_3$	$+0.5$	γ_3	$\gamma_0\ \gamma_3$
	-0.5	$\gamma_0\ \gamma_1$	γ_1	$+0.5$	$\gamma_0\ \gamma_1$	γ_1	$+0.5$	$\gamma_0\ \gamma_1$	γ_1	-0.5	$\gamma_0\ \gamma_1$	γ_1
	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$
	$+0.5$	$\gamma_0\ \gamma_2$	γ_2	-0.5	$\gamma_0\ \gamma_2$	γ_2	$+0.5$	$\gamma_0\ \gamma_2$	γ_2	-0.5	$\gamma_0\ \gamma_2$	γ_2
	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$
	$+0.5$	$\gamma_0\ \gamma_3$	γ_3	$+0.5$	$\gamma_0\ \gamma_3$	γ_3	-0.5	$\gamma_0\ \gamma_3$	γ_3	-0.5	$\gamma_0\ \gamma_3$	γ_3
	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$

Table 7.5: Different *static-light* operators in terms of *heavy-light* operators.

Combination 3		
$\frac{1}{2}(\mathcal{O}^{(\gamma_1+\gamma_0\gamma_1)\gamma_5}+\mathcal{O}^{(\gamma_2+\gamma_0\gamma_2)\gamma_5}$ $+\mathcal{O}^{(-\gamma_3-\gamma_0\gamma_3)\gamma_5}+\mathcal{O}^{(\mathbb{1}+\gamma_0)})$		
λ	Γ_1	Γ_2
$+1$	γ_5	γ_5
-1	γ_5	$\gamma_0\ \gamma_5$
$+1$	$\gamma_0\ \gamma_5$	γ_5
-1	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$

Table 7.6: Linear combination of four *static-light* operators that describe *heavy-light* operators with only η_c mesons.

Combination 4												
	$\mathcal{O}^{(\gamma_1-\gamma_0\gamma_1)\gamma_5}$			$\mathcal{O}^{(\gamma_2-\gamma_0\gamma_2)\gamma_5}$			$\mathcal{O}^{(-\gamma_3+\gamma_0\gamma_3)\gamma_5}$			$\mathcal{O}^{\mathbb{1}-\gamma_0}$		
Γ^1	$\mathcal{C}(\gamma_1-\gamma_0\gamma_1)\gamma_5$			$\mathcal{C}(\gamma_2-\gamma_0\gamma_2)\gamma_5$			$\mathcal{C}(-\gamma_3+\gamma_0\gamma_3)\gamma_5$			$\mathcal{C}(\mathbb{1}-\gamma_0)$		
Γ^2	$-\gamma_5(\gamma_1+\gamma_0\gamma_1)\mathcal{C}$			$-\gamma_5(\gamma_2+\gamma_0\gamma_2)\mathcal{C}$			$\gamma_5(\gamma_3+\gamma_0\gamma_3)\mathcal{C}$			$(\mathbb{1}-\gamma_0)\mathcal{C}$		
	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2	λ	Γ_1	Γ_2
	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5	$+0.5$	γ_5	γ_5
	$+0.5$	γ_5	$\gamma_0\ \gamma_5$	$+0.5$	γ_5	$\gamma_0\ \gamma_5$	$+0.5$	γ_5	$\gamma_0\ \gamma_5$	$+0.5$	γ_5	$\gamma_0\ \gamma_5$
	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5	-0.5	$\gamma_0\ \gamma_5$	γ_5
	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$	-0.5	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$
	-0.5	γ_1	γ_1	$+0.5$	γ_1	γ_1	$+0.5$	γ_1	γ_1	-0.5	γ_1	γ_1
	-0.5	γ_1	$\gamma_0\ \gamma_1$	$+0.5$	γ_1	$\gamma_0\ \gamma_1$	$+0.5$	γ_1	$\gamma_0\ \gamma_1$	-0.5	γ_1	$\gamma_0\ \gamma_1$
	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2	$+0.5$	γ_2	γ_2	-0.5	γ_2	γ_2
	$+0.5$	γ_2	$\gamma_0\ \gamma_2$	-0.5	γ_2	$\gamma_0\ \gamma_2$	$+0.5$	γ_2	$\gamma_0\ \gamma_2$	-0.5	γ_2	$\gamma_0\ \gamma_2$
	$+0.5$	γ_3	γ_3	$+0.5$	γ_3	γ_3	-0.5	γ_3	γ_3	-0.5	γ_3	γ_3
	$+0.5$	γ_3	$\gamma_0\ \gamma_3$	$+0.5$	γ_3	$\gamma_0\ \gamma_3$	-0.5	γ_3	$\gamma_0\ \gamma_3$	-0.5	γ_3	$\gamma_0\ \gamma_3$
	$+0.5$	$\gamma_0\ \gamma_1$	γ_1	-0.5	$\gamma_0\ \gamma_1$	γ_1	-0.5	$\gamma_0\ \gamma_1$	γ_1	$+0.5$	$\gamma_0\ \gamma_1$	γ_1
	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	-0.5	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$	$+0.5$	$\gamma_0\ \gamma_1$	$\gamma_0\ \gamma_1$
	-0.5	$\gamma_0\ \gamma_2$	γ_2	$+0.5$	$\gamma_0\ \gamma_2$	γ_2	-0.5	$\gamma_0\ \gamma_2$	γ_2	$+0.5$	$\gamma_0\ \gamma_2$	γ_2
	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	-0.5	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$	$+0.5$	$\gamma_0\ \gamma_2$	$\gamma_0\ \gamma_2$
	-0.5	$\gamma_0\ \gamma_3$	γ_3	-0.5	$\gamma_0\ \gamma_3$	γ_3	$+0.5$	$\gamma_0\ \gamma_3$	γ_3	$+0.5$	$\gamma_0\ \gamma_3$	γ_3
	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	-0.5	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$	$+0.5$	$\gamma_0\ \gamma_3$	$\gamma_0\ \gamma_3$

Table 7.7: Different *static-light* operators in terms of *heavy-light* operators.

Combination 4		
$\frac{1}{2}(\mathcal{O}^{(\gamma_1-\gamma_0\gamma_1)\gamma_5}+\mathcal{O}^{(\gamma_2-\gamma_0\gamma_2)\gamma_5}$ $+\mathcal{O}^{(-\gamma_3+\gamma_0\gamma_3)\gamma_5}+\mathcal{O}^{(\mathbb{1}-\gamma_0)})$		
λ	Γ^1	Γ^2
$+1$	γ_5	γ_5
$+1$	γ_5	$\gamma_0\ \gamma_5$
-1	$\gamma_0\ \gamma_5$	γ_5
-1	$\gamma_0\ \gamma_5$	$\gamma_0\ \gamma_5$

Table 7.8: Linear combination of four *static-light* operators that describe *heavy-light* operators with only η_c mesons.

If we now add all 16 *static-light* operators $\mathcal{O}_j$ with

$$\mathcal{O}_j \in \{\mathcal{O}^{\gamma_1+\gamma_0\gamma_1}, \mathcal{O}^{\gamma_2+\gamma_0\gamma_2}, \mathcal{O}^{\gamma_3+\gamma_0\gamma_3}, \mathcal{O}^{\gamma_5+\gamma_0\gamma_5}, \mathcal{O}^{\gamma_1-\gamma_0\gamma_1}, \mathcal{O}^{\gamma_2-\gamma_0\gamma_2}, \mathcal{O}^{\gamma_3-\gamma_0\gamma_3}, \mathcal{O}^{\gamma_5-\gamma_0\gamma_5}, \mathcal{O}^{(\gamma_1+\gamma_0\gamma_1)\gamma_5}, \mathcal{O}^{(\gamma_2+\gamma_0\gamma_2)\gamma_5}, \mathcal{O}^{(-\gamma_3-\gamma_0\gamma_3)\gamma_5}, \mathcal{O}^{(\mathbb{1}+\gamma_0)}, \mathcal{O}^{(\gamma_1-\gamma_0\gamma_1)\gamma_5}, \mathcal{O}^{(\gamma_2-\gamma_0\gamma_2)\gamma_5}, \mathcal{O}^{(-\gamma_3+\gamma_0\gamma_3)\gamma_5}, \mathcal{O}^{\mathbb{1}-\gamma_0}\}$$

and multiply it by a normalization factor, we get the expected result:

$\frac{1}{8}\sum_{j=1}^{16}\mathcal{O}_j$		
λ	Γ_1	Γ_2
$+1$	γ_5	γ_5

Table 7.9: $\bar{c}c\bar{c}c$ operator represented by 16 *static-light* operators.

The following tables are excerpts of an overview of all quantum numbers for different combinations of Γ^1 that have been computed for the studies of the *static-light* tetraquark.

	$\Gamma_X^{(ud\pm du)}$ tb	$\mathcal{P}^{(\mathrm{tm})}$, $\mathcal{P}_x^{(\mathrm{tm})}$, sec.	$\Gamma_X^{(ud\pm du)}$ ppb	$\mathcal{P}$, $\mathcal{P}_x$	type
$j_z = 0$, $I = 0$					
1	$\gamma_5^{(-)} - i\gamma_0^{(+)}$	$+, -, \quad a$	$(+\gamma_5 + \gamma_0\gamma_5)^{(-)}$	$-, +$	att SS
2	$\gamma_0\gamma_3^{(-)} - i\gamma_3\gamma_5^{(+)}$	$-, +, \quad b$	$(+\gamma_0\gamma_3 + \gamma_3)^{(-)}$	$+, -$	rep SS
$j_z = 0$, $I = 1$, $I_z = 0$					
3	$\gamma_0\gamma_3^{(+)} - i\gamma_3\gamma_5^{(-)}$	$-, -, \quad c$	$(+\gamma_0\gamma_3 + \gamma_3)^{(+)}$	$-, -$	att SS
4	$\gamma_5^{(+)} - i\gamma_0^{(-)}$	$+, +, \quad d$	$(+\gamma_5 + \gamma_0\gamma_5)^{(+)}$	$+, +$	rep SS
$j_z = 1$, $I = 0$					
5	$\gamma_0\gamma_{1/2}^{(-)} - i\gamma_{1/2}\gamma_5^{(+)}$	$-, -/+, \quad e/f$	$(+\gamma_0\gamma_{1/2} + \gamma_{1/2})^{(-)}$	$+, +/-$	rep SS
$j_z = 1$, $I = 1$, $I_z = 0$					
6	$\gamma_0\gamma_{1/2}^{(+)} - i\gamma_{1/2}\gamma_5^{(-)}$	$-, +/-, \quad f/e$	$(+\gamma_0\gamma_{1/2} + \gamma_{1/2})^{(+)}$	$-, +/-$	att SS

Table 7.10: Twisted and (pseudo) physical quantum numbers for $ud \pm du$. Taken from M.Wagner, to be published.

	$\Gamma_X^{(\frac{uu}{dd})}$ tb	$\mathcal{P}^{(\mathrm{tm})}\mathcal{P}_x^{(\mathrm{tm})}$, sec.	$\Gamma_X^{(\frac{uu}{dd})}$ ppb	$\mathcal{P}$, $\mathcal{P}_x$	type
$j_z = 0$, $I = 1$, $I_z = \pm$					
7	$\gamma_3 \pm i\gamma_0\gamma_3\gamma_5$	$+, \quad i$	$+\gamma_3 + \gamma_0\gamma_3$	$-, -$	att SS
8	$\gamma_0\gamma_5 \pm i$	$+, \quad i$	$+\gamma_0\gamma_5 + \gamma_5$	$+, +$	rep SS
$j_z = 1$, $I = 1$, $I_z = \pm$					
9	$\gamma_{1/2} \pm i\gamma_0\gamma_{1/2}\gamma_5$	$-/+, \quad k/l$	$+\gamma_{1/2} + \gamma_0\gamma_{1/2}$	$-, +/-$	att SS

Table 7.11: Twisted and (pseudo) physical quantum numbers for uu and dd. Taken from M. Wagner, to be published.

In order to understand the *heavy-light* operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ we analyse the *static-light* operators in the linear combination of table 7.9. Looking at table 7.10 and 7.11 we see which *static-light* operator defined by the γ-combinations in coloum 4 correspond to an attractive or repulsive potential.
Based on the representation of the $\bar{c}c\bar{c}c$ tetraquark candidate in the static limit, we find that this operator is actually a superposition of attractive and repulsive potentials for either $I = 0$ or $I = 1$. Switching the isospin from $I = 0$ to $I = 1$ results in turning an attractive potential into a repulsive and vice versa as shown in table 7.10 and 7.11.

Chapter 8

Numerical results and analysis for heavy-light 4-quark-states

As summarised in ch. 5.1, we expect to obtain different masses for the correlation functions with isospin states $I = 0$ and $I = 1$ in the static limit using the constructed *heavy-light* operators $\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$.
The results of the effective masses are given in the following sections.

8.1 *Heavy-light* operator $\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$

Computing the correlator of the operator given in eq. (6.3.8)

$$\begin{aligned}\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}} =\ & 0.5\,\cdot\,\bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(j)}\,\bar{c}(\gamma_5+\gamma_0\gamma_5)\,l^{(k)}\\ &+0.5\,\cdot\sum_{j=1}^{3}\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(j)}\,\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(k)}\end{aligned}$$

we obtain the following result for the effective masses:

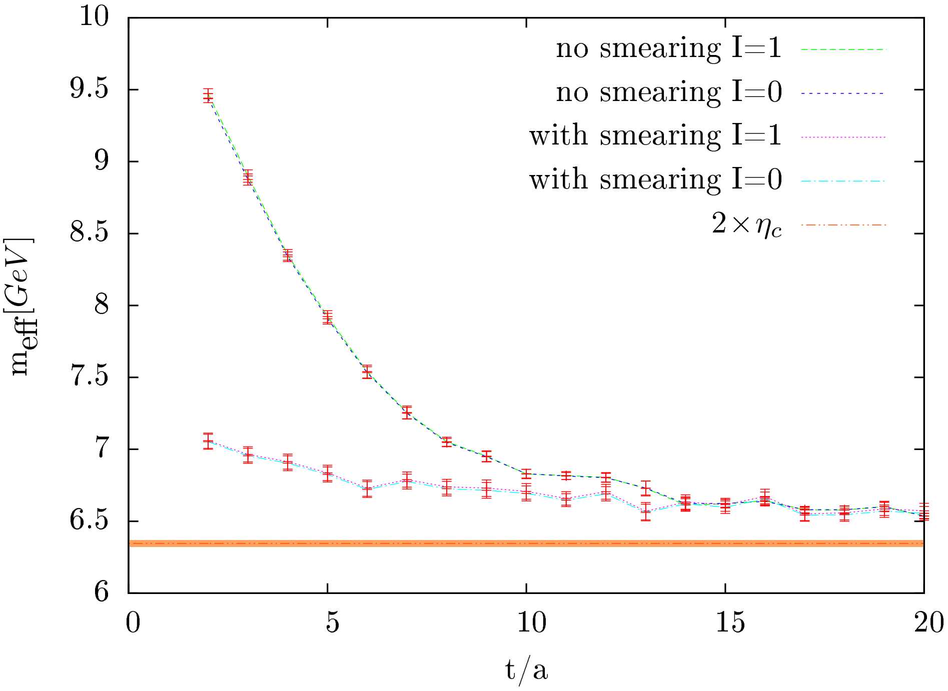


Figure 8.1: Effective masses for the isospin states $I = 0$ and $I = 1$ of the *heavy-light* operator in eq. (6.3.8) with and without smearing.

For the different isospin states using smearing techniques we obtain the following masses

$$\mathrm{m_{I=0}^{smea}} = (6.5754 \pm 0.0146)\ \mathrm{GeV}$$
$$\mathrm{m_{I=1}^{smea}} = (6.5897 \pm 0.0156)\ \mathrm{GeV}.$$

Calculating the differences between the masses as in ch. 5 we obtain

$$\mathrm{m_{I=1}^{smea}} - \mathrm{m_{I=0}^{smea}} = (0.0143 \pm 0.0017)\ \mathrm{GeV}.$$

In figure 8.1 we find again, that the operator without smearing does not represent the ground state because of the higher masses for small values of t.
We also find that for large values of t/a a clear plateau seems not to be reached.

However, we find a mass difference of about 14 MeV with a confidence level of $\approx 8\sigma$. Even if this mass difference is smaller than the mass difference obtained in ch. 5.2, this might be a first indication for having an attractive and repulsive channel. Looking at table 7.10 we also find that in the static limit the isospin state $I = 0$ leads to an attractive channel for combining the light quarks with $\gamma_5 + \gamma_0\gamma_5$, cf. row number 1. This is in accordance with our result of the effective masses which indicates that the isospin state $I = 0$ might be an attractive channel.

8.2 *Heavy-light* operator $\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$

Computing the correlation function of the operator given in eq. (7.0.2)

$$\begin{aligned}\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}} &= \bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(j)}\;\bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(k)}\\ &-\frac{1}{3}\sum_{j=1}^{3}\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(j)}\;\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(k)}\end{aligned}$$

yield to the following effective masses:

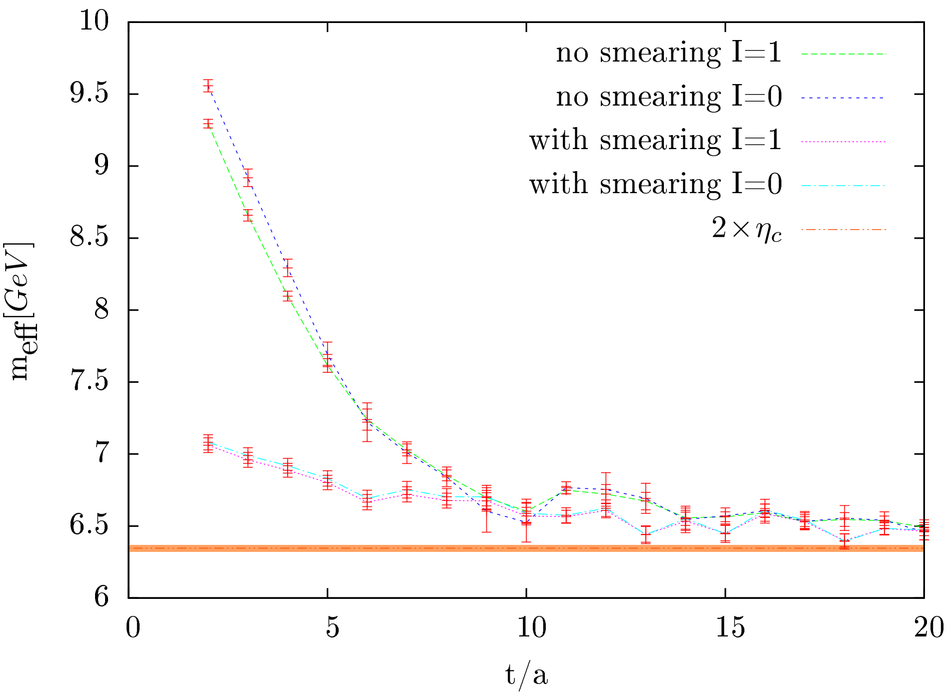


Figure 8.2: Results for the isospin states $I = 0$ and $I = 1$ of *heavy-light* operator in eq. (7.0.2) with and without smearing.

For the masses we find the following result:

$$\begin{aligned}\mathrm{m}_{\mathrm{I=0}}^{\mathrm{smea}} &= (6.4779 \pm 0.0197)\ \mathrm{GeV}\\ \mathrm{m}_{\mathrm{I=1}}^{\mathrm{smea}} &= (6.4724 \pm 0.0167)\ \mathrm{GeV}\end{aligned}$$

Again, calculating the difference of the two masses leads to

$$\mathrm{m}_{\mathrm{I=0}}^{\mathrm{smea}} - \mathrm{m}_{\mathrm{I=1}}^{\mathrm{smea}} = (0.0054 \pm 0.0039)\ \mathrm{GeV}.$$

Since we do not find a clear plateau for large values of t/a and the uncertainty of the difference of the masses is of the same magnitude as the difference itself, we do not observe a clear indication of different masses for the two isospin states. However, we find a lower mass for the isospin state $I = 1$, which is also in accordance with the attractive channels given in row 3 and 6 of table 7.10.

8.3 Conclusion of ch. 8.1 and 8.2

In figure 8.1 and 8.2 we see, that the operators do not represent a ground state, but rather a mixture with excited states. This is indicated by the still decreasing masses for large time separations, which imply, that the higher order terms in eq. (3.2.2) do not vanish.

Computing the effective masses for the different isospin states $I = 0$ and $I = 1$ of the constructed operator $\mathcal{O}_1^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ we find a slight difference of about 14 MeV with a confidence level of $\approx 8\sigma$. The lower mass correspond to the isospin state $I = 0$ which is in accordance with the result obtained in the static limit, cf. row 1 in table 7.10.

By computing the correlator with the constructed operator $\mathcal{O}_2^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ we obtain an uncertainty of the difference of the masses for $I = 0$ and $I = 1$ of the same magnitude as the difference itself. Thus, we do not observe a clear signal for two different masses. However, it is indicated that we obtain a smaller mass for the isospin state $I = 1$ which is again in accordance with computations in the static limit, cf. row 3 and 6 in table 7.10.
Even if we do not find a significant evidence for different masses, this might be a first indication and need to be investigated in more detail.
Although, the operators $\mathcal{O}_1^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ do not approximate the ground state for the $\bar{c}c\bar{c}c$ 4-quark-state, they might be good candidates for finding a signal of bound states in the static limit, when using the masses of up and down quarks for the computations. By using different quark masses, we expect to find different masses for the quantum numbers $I = 0$ and $I = 1$.

Using the operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ for computing the effective mass we find a significant difference of ≈ 60 MeV, cf. ch. 5.2. Thus, we assume that this operator excites the ground state in the non-static regime.

Chapter 9

Comparison of *heavy-light* operators and simple model of interpretation

9.1 Comparison of numerical results for $\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{\gamma_5,\gamma_5}$, $\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{1}$ and $\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{2}$

Comparing the effective masses of the *heavy-light* operator of eq. (5.2.1)

$$\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{\gamma_5,\gamma_5} = \bar{c}_A(\gamma_5)_{AB}\, l^{(j)}_B\, \bar{c}_C(\gamma_5)_{CD}\, l^{(k)}_D.$$

with the effective masses of the *heavy-light* operator given in eq. (6.3.8)

$$\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{1} = 0.5\cdot\bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(j)}\,\bar{c}(\gamma_5+\gamma_0\gamma_5)\,l^{(k)}+0.5\cdot\sum_{j=1}^{3}\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(j)}\,\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(k)},$$

and in eq. (7.0.2)

$$\mathcal{O}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}_{2} = \bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(j)}\,\bar{c}\,(\gamma_5+\gamma_0\gamma_5)\,l^{(k)} - \frac{1}{3}\sum_{j=1}^{3}\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(j)}\,\bar{c}\,(\gamma_j+\gamma_0\gamma_j)\,l^{(k)}$$

using APE and Gaussian smearing leads to the following figures:

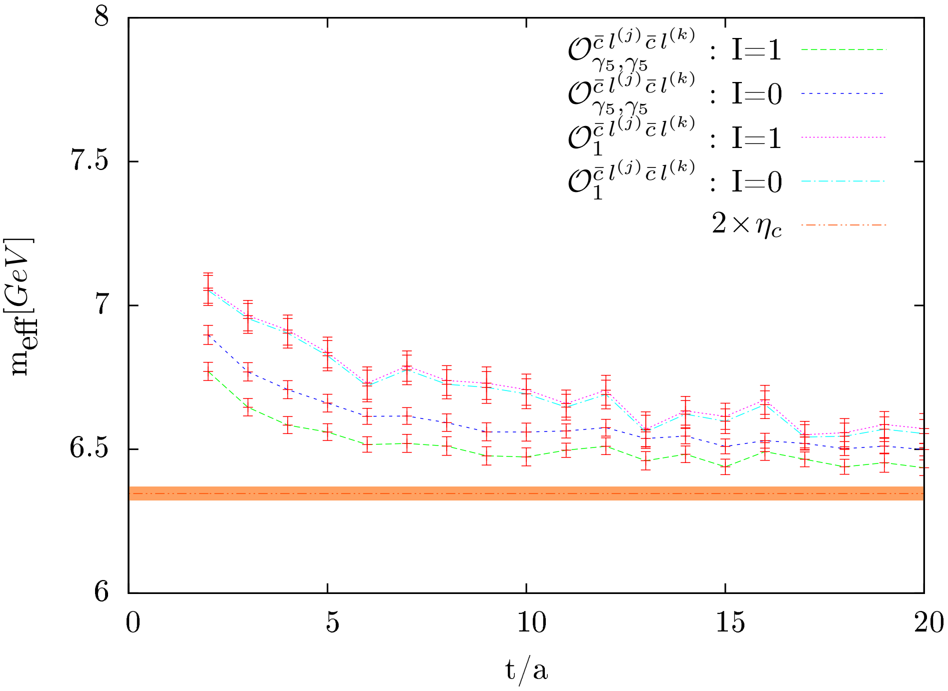


Figure 9.1: Comparing the effective masses of the correlators using the operators of eq. (5.2.1) and of eq. (6.3.8) using smearing techniques.

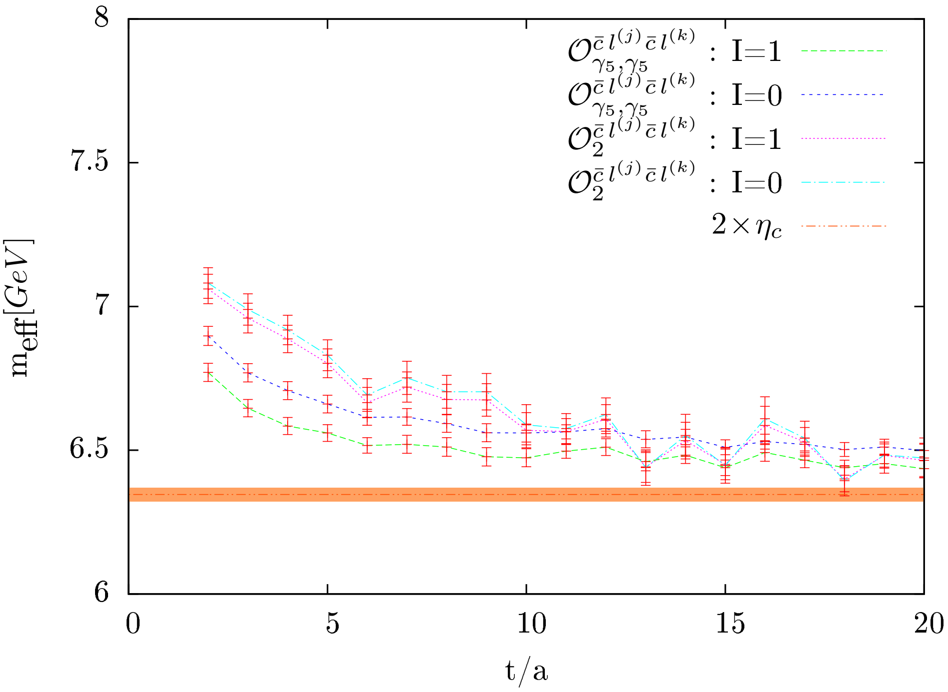


Figure 9.2: Comparing the effective masses of the correlators using the operators of eq. (5.2.1) and of eq. (7.0.2) using smearing techniques.

In figure 9.1 and 9.2 we see, that the *heavy-light* operator $\mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}_{\gamma_5,\gamma_5}$ consisting of only η_c mesons leads to a better overlap with the ground state than the *heavy-light* operators $\mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}_{1}$ and $\mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}_{2}$ which do also include J/Ψ mesons because of the lower mass. The operators $\mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}_{1}$ and $\mathcal{O}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}_{2}$ do not yield to conclusive results, because for $t \leqq 20$ the effective masses seem not to have reached a plateau yet.

9.2 Concept for analysing the results obtained in ch. 9.1

In order to illustrate some basic concepts about the contribution of the ground state and excited states to the effective mass let us consider the Hamiltonian H_{total} of our states in a simple quantum mechanical model.
In general it holds, that

$$H_{\text{total}} = H_{\text{free}} + H_{\text{int}}$$

with H_{free} denoting the Hamiltonian without any interaction and H_{int} denoting the contribution of an interaction.
More precisely, we define

$$H_{\text{free}} = \begin{pmatrix} m & \\ & M \end{pmatrix} \text{ in the basis of } \begin{pmatrix} \text{“}2 \times \eta_c\text{”} \\ \text{“}2 \times J/\Psi\text{”} \end{pmatrix} \tag{9.2.1}$$

$$H_{\text{int}} = \begin{pmatrix} +\Delta & \\ & -\epsilon \end{pmatrix} \text{ in the basis of } \begin{pmatrix} \text{“}2 \times \eta_c + 2 \times J/\Psi\text{”} \\ \text{“}2 \times \eta_c - 2 \times J/\Psi\text{”} \end{pmatrix} \tag{9.2.2}$$

In our case the Hamiltonian H_{free} denotes the Hamiltonian of the two-meson-state with $m = 2 \times m_{\eta_c}$ and $M = 2 \times m_{J/\Psi}$, i.e. the Hamiltonian for two non-interacting mesons. H_{int} however denotes the Hamiltonian of the meson-meson interaction. Analogously to the constructed operators of eq. (6.3.8) and (7.0.1) , we can have either the state of $2 \times \eta_c + 2 \times J/\Psi$ or the state of $2 \times \eta_c - 2 \times J/\Psi$ with the masses Δ and ϵ respectively. There are different signs for the two states, indicating an attractive and repulsive potential.

In order to obtain an expression for the Hamiltonian H_{total}, we need to transform the basis of H_{int} into the basis of H_{free} with the following transformation matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \text{“}2 \times \eta_c\text{”} \\ \text{“}2 \times J/\Psi\text{”} \end{pmatrix} = \begin{pmatrix} \text{“}2 \times \eta_c + 2 \times J/\Psi\text{”} \\ \text{“}2 \times \eta_c - 2 \times J/\Psi\text{”} \end{pmatrix} \tag{9.2.3}$$

Thus, we obtain

$$H_{\text{int}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}+\Delta & \\ & -\epsilon\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{9.2.4}$$

$$= \frac{1}{2}\begin{pmatrix}\Delta-\epsilon & \Delta+\epsilon\\ \Delta+\epsilon & \Delta-\epsilon\end{pmatrix} \tag{9.2.5}$$

$$= \frac{1}{2}\begin{pmatrix}\Delta-\epsilon & 0\\ 0 & \Delta-\epsilon\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & \Delta+\epsilon\\ \Delta+\epsilon & 0\end{pmatrix} \tag{9.2.6}$$

$$= H_{\text{int,shift}} + H_{\text{int, split}} \tag{9.2.7}$$

For the Hamiltonian H_{total} we obtain

$$H_{\text{total}} = H_{\text{free}} + H_{\text{int}} = \begin{pmatrix}m + \frac{\Delta-\epsilon}{2} & \frac{\Delta+\epsilon}{2}\\ \frac{\Delta+\epsilon}{2} & M + \frac{\Delta-\epsilon}{2}\end{pmatrix} \tag{9.2.8}$$

$$= \begin{pmatrix}m' & \delta\\ \delta & M'\end{pmatrix}$$

$$= \begin{pmatrix}m' & 0\\ 0 & M'\end{pmatrix} + \begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix} \tag{9.2.9}$$

$$= H_{\text{free}} + H_{\text{int,shift}} + H_{\text{int, split}} \tag{9.2.10}$$

with $m' = m + \frac{\Delta-\epsilon}{2}$ and $M' = M + \frac{\Delta-\epsilon}{2}$ and $\delta = \frac{\Delta+\epsilon}{2}$.

Note, that the interaction term H_{int} of the Hamiltonian H_{total} is now represented in the same basis as H_{free}.

Now, we need to find the eigenvalues and eigenvectors of the Hamiltonian H_{total}.

$$0 = \det(H_{\text{total}} - \mathbb{1}\lambda) = (m' - \lambda)(M' - \lambda) - \delta^2 \tag{9.2.11}$$

Solving eq. (9.2.11) leads to

$$\lambda_{1/2} = \frac{m' + M'}{2} \pm \sqrt{\left(\frac{m' - M'}{2}\right)^2 + \delta^2} \tag{9.2.12}$$

$$= \frac{m + M}{2} + (\Delta - \epsilon) \pm \sqrt{\left(\frac{m - M}{2}\right)^2 + \delta^2} \tag{9.2.13}$$

Now, we need to consider two cases:

- Case 1: $\delta \ll |m' - M'|$

  With a taylor expansion for $\delta \ll |m' - M'|$, i.e., a large difference of the masses and a small binding energy δ, we find the eigenvalues

$$\begin{aligned}\lambda_{1/2} &= \frac{m'+M'}{2} \pm \frac{m'-M'}{2} \cdot \left(1 + \frac{2\delta^2}{(m'-M')^2}\right) \\ &= \frac{m+M}{2} + (\Delta - \epsilon) \pm \frac{m-M}{2} \cdot \left(1 + \frac{2(\Delta+\epsilon)^2}{(m-M)^2}\right).\end{aligned} \tag{9.2.14}$$

Assuming $\delta = \Delta + \epsilon \ll (m - M)$, we obtain the eigenvectors

$$\begin{aligned} x_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{“}2 \times \eta_c\text{”} \\ x_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{“}2 \times J/\Psi\text{”}. \end{aligned}$$

With $\delta \approx 0$ we obtain

$$\begin{pmatrix} m' & 0 \\ 0 & M' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = m' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(m + \frac{\Delta - \epsilon}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9.2.15}$$

$$\begin{pmatrix} m' & 0 \\ 0 & M' \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = M' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(M + \frac{\Delta - \epsilon}{2}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.2.16}$$

With eq. (9.2.15) and (9.2.16) we find, that our eigenstate is either in the basis of $2 \times \eta_c$ with the eigenvalue m' or $2 \times J/\Psi$ with the eigenvalue M', i.e., a state of two non-interacting mesons.

- Case 2: $|m' - M'| \ll \delta$
  With a taylor expansion for $|m' - M'| \ll \delta$ we obtain

$$\lambda_{1/2} = \frac{m+M}{2} + (\Delta - \epsilon) \pm \delta \cdot \left(1 + \frac{(m-M)^2}{8\delta^2}\right) \tag{9.2.17}$$

and the corresponding eigenvectors

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{“}2 \times \eta_c + 2 \times J/\Psi\text{”} \\ x_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{“}2 \times \eta_c - 2 \times J/\Psi\text{”}. \end{aligned}$$

Assuming $m' - M' \approx 0 \rightarrow m' \approx M'$ we obtain

$$\begin{pmatrix} m' & \delta \\ \delta & M' \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(m' + \delta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(m + \Delta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9.2.18}$$

$$\begin{pmatrix} m' & \delta \\ \delta & M' \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(m' - \delta) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(m - \epsilon) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{9.2.19}$$

Eq. (9.2.18) and (9.2.19) show, that we have either an eigenstates in the basis of $2\times\eta_c + 2\times J/\Psi$ or $2\times\eta_c - 2\times J/\Psi$ with the eigenvalues $m+\Delta$ and $m-\epsilon$ respectively.

9.3 Conclusion of ch. 9.1 and 9.2

Based on the previous considerations, we have

- **Case 1:** $m - M \gg \delta \rightarrow$ state of either $2 \times \eta_c$ or $2 \times J/\Psi$
- **Case 2:** $m - M \ll \delta \rightarrow$ state of either $2 \times \eta_c + 2 \times J/\Psi$ or $2 \times \eta_c - 2 \times J/\Psi$

Depending on the case that we are considering, we have either a small δ or a small difference of the masses $m - M$.

In the first case, we have an eigenstate of η_c or J/Ψ mesons with a small interaction $\delta \approx 0$. In the second case, the eigenstates are a superposition of two η_c and two J/Ψ mesons.

Note, that the term $\Delta - \epsilon \neq 0$ due to the interaction leads to a shift of the energy levels. This shift might have been observed by the plots in figure 5.1,9.1 and 9.2 by the difference of the mass of two η_c mesons and the $\bar{c}c\bar{c}c$ 4-quark-states. Thus, we do not obtain exactly the mass of two η_c mesons.

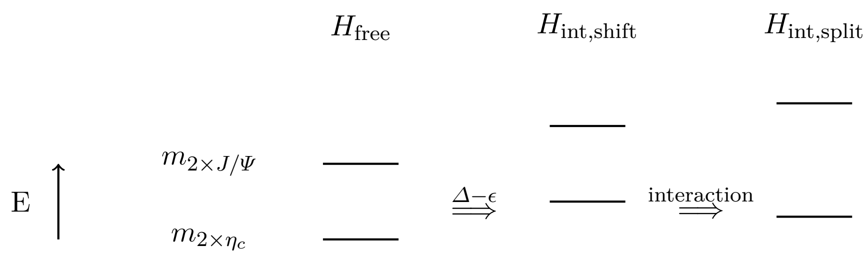


Figure 9.3: Visualisation of energy shift.

With the results of figure 9.1 and 9.2 we find, that

- the quark flavours and masses used describe the regime of $\delta \ll |m' - M'|^*$, which is indicated by [14],
- the operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ consisting of η_c mesons leads to a better excitation of the ground state than the operators $\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ consisting of η_c and J/Ψ mesons.

Since the operators $\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ do not lead to conclusive results, we can also assume, that

- the mass of the charm quarks might not be heavy enough for approximating static quarks,
- we need the mass of an up or down quark for the light quarks.

*If the operators would describe the case of $|m' - M'| \ll \delta$, then we would have obtained the opposite result. The operators $\mathcal{O}_1^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$ would have led to smaller masses than the operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}\,l^{(j)}\bar{c}\,l^{(k)}}$.

Chapter 10

Summary and outlook

In this work we investigated effective masses by computing correlation functions of different 4-quark operators using lattice QCD. For the computations we used the twisted mass formulation and different smearing techniques. To save CPU time we set all quark masses equal to the charm quark mass.

First, we analysed the dynamic $\bar{c}c\bar{c}c$ tetraquark-candidate. However, we did not find a clear signal for a bound state in contrast to the result based given in [3].

Previous investigations of *static-light* 4-quark-states led to attractive interactions between two mesons and even bound tetraquark states [14]. Thus, we considered an *heavy-light* approximation of the static limit. In this approximation two heavy charm anti-quarks and two light quarks, i.e., up or down, were used.

Computing the correlation function of the *heavy-light* operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$, corresponding to the dynamic $\bar{c}c\bar{c}c$ tetraquark-candidate, led to a significant difference of the masses for the isospin states $I = 0$ and $I = 1$. In analogy to the investigations of *static-light* 4-quark-states, we assume to have found attractive and repulsive channels.

Additionally, we expressed *static-light* operators in terms of *heavy-light* operators and computed the corresponding correlation functions of the constructed *heavy-light* operators $\mathcal{O}_1^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$. For these operators slightly different masses were obtained for the isospin states $I = 0$ and $I = 1$, which give a first indication of interactions. However the numerical results and a simple quantum mechanical model showed, that the operator $\mathcal{O}_{\gamma_5,\gamma_5}^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ describes the ground state in the non-static regime, whereas the operators $\mathcal{O}_1^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ might excite the ground state in the static limit.

In order to improve our investigations and to verify our assumptions, it will be necessary to use different quark masses for the computations, i.e., the masses of an up or down quark and a higher mass for the anti-quarks instead of using 4 charm-quark masses and to increase the statistical accuracy. Then, we might be able to find attractive channels or even bound states for the constructed *heavy-light* operators $\mathcal{O}_1^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$ and $\mathcal{O}_2^{\bar{c}l^{(j)}\bar{c}l^{(k)}}$.

Acknowledgements

At this point I would like to thank several people for supporting me during the work of this master thesis.

First and most of all I want to thank Marc Wagner for his exceptional support and excellent supervision.

I also want to thank David Palao and Joshua Berlin for their helpful discussions.

Additionally, special thanks go to my fellow students and friends, Constantin Butzke, Björn Wagenbach and Philipp Wolf, who have always been supporting me during the last months and years.

Bibliography

[1] Roel Aaij et al. Observation of the resonant character of the $Z(4430)^{-}$ state. *Phys.Rev.Lett.*, 112:222002, 2014.

[2] M.F.M. Lutz et al. Physics Performance Report for PANDA: Strong Interaction Studies with Antiprotons. 2009.

[3] Walter Heupel, Gernot Eichmann, and Christian S. Fischer. Tetraquark Bound States in a Bethe-Salpeter Approach. *Phys.Lett.*, B718:545–549, 2012.

[4] Marc Wagner. Forces between static-light mesons. *PoS*, LATTICE2010:162, 2010.

[5] Marc Wagner. Static-static-light-light tetraquarks in lattice QCD. *Acta Phys.Polon.Supp.*, 4:747–752, 2011.

[6] B. Wagenbach. Master's thesis.

[7] Y. Iwasaki. *Renormalization group analysis of lattice theories and improved lattice action: two-dimensional non-linear O(N) sigma model.* Nucl. Phys. B 258, 1985.

[8] R. Baron, Ph. Boucaud, J. Carbonell, A. Deuzeman, V. Drach, et al. Light hadrons from lattice QCD with light (u,d), strange and charm dynamical quarks. *JHEP*, 1006:111, 2010.

[9] Roberto Frezzotti, Pietro Antonio Grassi, Stefan Sint, and Peter Weisz. Lattice QCD with a chirally twisted mass term. *JHEP*, 0108:058, 2001.

[10] R. Frezzotti and G.C. Rossi. Twisted mass lattice QCD with mass nondegenerate quarks. *Nucl.Phys.Proc.Suppl.*, 128:193–202, 2004.

[11] Constantia Alexandrou, Jan Oliver Daldrop, Mattia Dalla Brida, Mario Gravina, Luigi Scorzato, et al. Lattice investigation of the scalar mesons $a_0(980)$ and κ using four-quark operators. *JHEP*, 1304:137, 2013.

[12] J. Berlin. Master's thesis. 2013.

[13] C. Wiese. Master's thesis. 2012.

[14] Pedro Bicudo and Marc Wagner. Lattice QCD signal for a bottom-bottom tetraquark. *Phys.Rev.*, D87(11):114511, 2013.

[15] Karl Jansen, Chris Michael, Andrea Shindler, and Marc Wagner. The Static-light meson spectrum from twisted mass lattice QCD. *JHEP*, 0812:058, 2008.

[16] Particle physics booklet extracted from the review of particle physics j. beringer et al. (particle data group), phys. rev. d 86, 010001 (2012).

[17] http://www.physics.utah.edu/ detar/phycs6730/handouts/jackknife/jackknife/jackknife.html.

[18] L. B. Okun. Leptons and Quarks. 1980.