Noncommutative Gravity and Solitons

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- Gauge theories on NC spaces: in string theory NC U(n) YM in D=p+1 as low energy limit of n Dp-branes in backround B field

- Many solutions to classical field equations, e.g.

  - instantons in D=4+0
  - monopoles and strings in D=3+1

- NC scalar field theory likewise arises in low energy limit of string theory

  Derrick theorem circumvented; solitons, first constructed by Gopakumar et al., can be interpreted as D-brane solitons of open string field theory tachyon

- From the field theory viewpoint, gauge theories on NC spaces can be formulated alternatively

  - in terms of ordinary fields multiplied with the ★product

  - in the operator formalism in which all fields are operators in a suitable Hilbert space

  This is the Moyal-Weyl correspondence

- The MW correspondence is 1-1, however consistency of the ★ product requires G=U(n) or certain subgroups thereof

- MW plane and MW correspondence: consider flat NC space-time with coordinates (x,y,t) with t a real commuting parameter and x,y represented by operators $\hat{x}, \hat{y}$ satisfying the CR $[\hat{x}, \hat{y}] = -i\theta$

  θ commuting parameter of length dimension 2

  - To an ordinary function f(x,y) with Fourier decomposition

$$f(x,y) = \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k) e^{i(k_x x + k_y y)}$$

is associated the Weyl ordering of the operator

$$\hat{f}(\hat{x},\hat{y}) = \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k) e^{i(k_x \hat{x} + k_y \hat{y})}$$

- The operator product $\hat{f} \cdot \hat{g}$ of two operators $\hat{f}, \hat{g}$ corresponds to the ★ product of functions f, g defined by

$$f \star g(x,y) \equiv e^{-\frac{i}{2}\theta\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2}\right)} f(x_1, y_1) g(x_2, y_2)|_{x_1=x_2, y_1=y_2}$$

- A basis for the Hilbert space is given by the eigenstates $\{|n\rangle\}$ of creation and annihilation operators defined by

$$\hat{a} = \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}} \qquad \hat{a}^\dagger = \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}}$$

and satisfying the CR relations $[\hat{a}, \hat{a}^\dagger] = 1$

- The projection operator $|n\rangle\langle n|$ is mapped to the function $2(-1)^n L_n(\frac{2r^2}{\theta})e^{-\frac{r^2}{\theta}}$

- Under the Moyal-Weyl mapping, integration over space becomes the operator trace, which we denote by Tr:

$$\int d^2x f \longleftrightarrow 2\pi\theta \mathrm{Tr} \hat{f}$$

- Derivatives are related to commutators as follows:

$$\partial_x f \to -\frac{i}{\theta}[\hat{y}, \hat{f}] \qquad\qquad \partial_y f \to \frac{i}{\theta}[\hat{x}, \hat{f}]$$

- Noncommutative field theory Lagrangians can be written in the operator framework or in terms of commutative fields for which the ordinary product is replaced by the ★ product and the two approaches are related by the above rules and definitions

- What about NC gravity? Various theories were considered, including in particular theories obtained

  - by gauging U(1,D-1) Lorentz groups instead of the standard SO(1,D-1) ones

  - by deforming the group of diffeomorphisms

  - by making use of the relationship between pure Einstein gravity and Chern-Simons theory in D=3

- Such theories do not arise as simple limits in string theory, however they may be viewed as models of physics at the Planck scale

- Several explicit metrics have been constructed perturbatively in the NC parameter θ, including NC corrections to black holes
- This talk: can one construct NC gravity solutions which are non-perturbative in θ (and hence have no commutative counterparts)?
- This question was recently investigated by Asakawa & Kobayashi in the context of a D=3 theory with the following action

$$S = -\frac{\Lambda}{\kappa^2} \int dt d^2 x E^{\star}$$

Λ: cosmological constant, ϰ: gravitational coupling,

$$E^{\star} = \det_{\star} E^a_\mu = \frac{1}{3!} \epsilon^{\mu\nu\rho} \epsilon_{abc} E^a_\mu \star E^b_\nu \star E^c_\rho$$

$E^a_\mu(x)$: dreibein

Field equations:

$$\epsilon^{\mu\nu\rho} \epsilon_{abc} \{E^b_\mu, E^c_\nu\}_{\star} = 0$$

Explicit solutions of A&K:

i) $E^a_\mu = 0$ trivial

ii) $$E^a_\mu = \begin{pmatrix} \alpha_0\phi_0 & & \\ & \alpha_1\phi_1 & \\ & & \alpha_2\phi_2 \end{pmatrix}$$ projectors

$$\phi_i \star \phi_j = \delta_{ij}\phi_i \qquad \Sigma_i \phi_i = 1$$

iii) $$E^a_\mu = \begin{pmatrix} \gamma^0 & & \\ & \gamma^1 & \\ & & \gamma^2 \end{pmatrix}$$ Clifford algebras

$$\{\gamma^a, \gamma^b\}_\star = 2\delta^{ab}$$

NC metric: $$G_{\mu\nu} = \frac{1}{2}\left(E^a_\mu \star E^b_\nu + E^b_\nu \star E^a_\mu\right)\eta_{ab}$$

- All solutions of type ii) have

$$\det_{\star} G_{\mu\nu} = 0$$

and are degenerate

All solutions of type iii) have

$$\det_{\star} G_{\mu\nu} \neq 0$$

and were interpreted by A&K as interpolations between the degenerate metric

$$G_{\mu\nu} = 0$$

near the origin of the Moyal plane and the Minkowski metric $G_{\mu\nu} = \eta_{\mu\nu}$ at infinity

- A&K solutions inspired by solitons in NC scalar field theory of Gopakumar et al.

$$S = \int dt d^2x \, (\partial_\mu \phi \star \partial^\mu \phi + \lambda \phi \star \phi \star \phi)$$

In the $\theta \to \infty$ limit, kinetic term is negligible compared to the potential term and solutions are easily found using projectors

The idea of A&K is to view their results as the gravitational analog of this, with the generalised determinant $-\frac{\Lambda}{\kappa^2} E^\star(x)$

playing the role of a potential term constructed from the dreibein $E^a_\mu(x)$ thought of as a 3X3 operator- valued matrix

- At finite θ, the original Gopakumar et al. solutions are only approximate, however higher order corrections in a 1/θ expansion were later constructed systematically by Gopakumar, Headrick & Spradlin and exact nonpertubative solutions were found by Harvey, Kraus & Larsen

- The gravitational analog of this situation would be to construct approximate and exact solutions of a theory of the form

$$S = -\frac{1}{\kappa^2} \int dt d^2x E^\star \star (R_\star - 2\Lambda)$$

for a suitably defined (e.g. Wess et al.) scalar curvature $R_\star$

- Can one construct exact solutions of this theory at finite θ?
  Apparent problems:
  - the "potential" $E^\star(x)$ is a complicated functional of all 9 components of $E^a_\mu(x)$
  - $E^\star(x)$ is not bounded from below and it is not clear whether it admits a pair of extrema between which stable solitons could interpolate
- Idea: sidestep these issues and construct exact solutions in a Chern-Simons type theory of NC D=3 gravity generalizing the well-known equivalence of pure Einstein gravity in D=3 and SO(2,1) Chern-Simons

- Consider NC Chern-Simons theory with gauge group G and action

$$S_{CS} = \int dt 2\pi\theta Tr \left[\frac{\kappa}{2} \epsilon^{\mu\nu\rho} tr_G \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right]$$

The coupling κ is related to the CS level k by $\kappa = \dfrac{k}{2\pi}$

- This is invariant under NC gauge transformations:

$$A_\mu \to U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

for suitable U and κ is quantised if $\Pi_1(G) \neq 0$

e.g. for G=U(n) and even for NC G=U(1) Chern-Simons (Nair&Polychronakos)

- Now take $G = U(1,1)_\star \times U(1,1)_\star$ with the same level k in both factors

  Let A and Ã denote the gauge fields for each factor
  A single CS breaks parity, however the difference

  $$S = S_{CS} - \tilde{S}_{CS}$$

  is parity invariant, as gravity is

  The choice of G is natural for a NC generalisation of the relationship between D=3 gravity and SO(2,1)xSO(2,1) Chern-Simons theory in Lorentzian spacetime

- Can choose all field to be real and the reality property is preserved under NC gauge transformations

- Writing

$$A = \omega + \frac{1}{l} e + \frac{i}{2} b$$

$$\tilde{A} = \omega - \frac{1}{l} e + \frac{i}{2} \tilde{b}$$

and substituting in the action yields (Cacciatori, Klemm, Martucci & Zanon)

$$S_{ECS} = \frac{1}{l} \int \epsilon_{abc} \left(e^a \wedge R^{bc} - \frac{1}{3l^2} e^a \wedge e^b \wedge e^c \right)$$

$$- \frac{1}{2} \int (b \wedge db - b \wedge b \wedge b) + \frac{1}{2} \int \left(\tilde{b} \wedge d\tilde{b} - \tilde{b} \wedge \tilde{b} \wedge \tilde{b} \right)$$

$$+ \frac{i}{2l} \int \eta_{ab} \left(e^a \wedge \omega^b + \omega^a \wedge e^b \right) \wedge \left(b + \tilde{b} \right)$$

$$+ \frac{i}{2l} \int \eta_{ab} \left(\omega^a \wedge \omega^b + \frac{1}{l^2} e^a \wedge e^b \right) \wedge \left(b - \tilde{b} \right)$$

All wedge products are taken with respect to the ★ product and $\mathcal{R} = d\omega + \omega \wedge \omega \qquad \mathcal{T} = de + \omega \wedge e + e \wedge \omega$

Interpreting ω as the Levi-Civita connection and expanding in a basis of $U(1,1)_\star$ one identifies NC generalisations of the familiar expressions for the curvature and torsion:

$$R^a = d\omega^a - \frac{1}{2}\epsilon^a{}_{bc}\omega^a \wedge \omega^b$$

$$T^a = de^a + \frac{1}{2}\left(\omega^a{}_b \wedge e^b + \omega^b \wedge \omega_b{}^a\right)$$

The theory thus describes a D=3 theory of torsionful NC gravity coupled to two NC abelian gauge fields b, ~b

- The invariance under NC gauge transformations translates into transformations of the fields $\omega, e, b, \tilde{b}$ which leave the action invariant and can be interpreted as deformed rotations, translations and U(1)xU(1) gauge transformations

- Now consider adding matter to the NS Chern-Simons action
  In order to study vortices and solitons in our model, we mostly need to consider charged NC Higgs fields with a particular sixth order potential

$$S_{\rm Higgs} = \int dt 2\pi\theta {\rm Tr}{\rm tr}_G \left[D_\mu \phi D^\mu \phi^\dagger + V(\phi\phi^\dagger)\right]$$

- Parity invariance requires that ϕ transforms as

$$\phi \rightarrow \phi^\dagger$$

The charged field ϕ can be chosen to transform in e.g. the fundamental, the adjoint or the bifundamental representation of $U(1,1)_\star \times U(1,1)_\star$

e.g.

$$\phi \rightarrow U \phi \tilde{U}$$

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \phi + i \phi \tilde{A}_\mu$$

- Take the Higgs potential to be

$$V(\xi) = -\frac{1}{\kappa^2} \xi \left(v^2 - \xi \right)^2$$

This is the Bogomol'nyi limit of the general gauge invariant and renormalisable potential possessing a symmetry breaking minimum at $v = |\phi|$

In this limit the symmetric minimum is degenerate with the asymmetric on

- The coupling of ϕ to the gravity, torsion and CS gauge fields is obtained by writing $\phi = \phi^a \tau_a + \varphi$ and expanding the Higgs action

- So the action for the model can be written alternatively as

$S(A, \tilde{A}, \phi)$ "Chern-Simons" description

$S(\omega, e, b, \tilde{b}, \phi^a, \varphi)$ "gravitational" description

- Aside: it is interesting to compare the $\theta \to 0$ limit of the model to the bosonic part of the ABJM theory, which was proposed as a description of N M2 branes probing a C^4/Z_R singularity

  More precisely, one needs to compare with the bosonic part of the maximally SUSY mass deformation of ABJM constructed by Gomis et al.

- One finds that the $\theta \to 0$ limit of the U(2)xU(2) Chern-Simons-Higgs model can be identified with the bosonic sector of the mass deformed ABJM model (with a single scalar in the bifundamental representation turned on)
  So it can be viewed as a toy model for a NC generalisation of the U(2)xU(2) mass deformed ABJM model and it would be interesting to study possible N=6 supersymmetric generalisations as well as relevant brane constructions with B fields

- For the time being we we wish to derive and solve the BPS equations of the NC model with action:

$$S = \int dt 2\pi\theta \mathrm{Tr} \frac{\kappa}{2} \Big[\epsilon^{\mu\nu\rho} \mathrm{tr} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right)$$

$$- \epsilon^{\mu\nu\rho} \mathrm{tr} \left(\tilde{A}_\mu \partial_\nu \tilde{A}_\rho - \frac{2i}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) \Big]$$

$$+ \int dt 2\pi\theta \mathrm{Tr}\mathrm{tr} \left[D_\mu \phi (D^\mu \phi)^\dagger + V(|\phi|^2) \right]$$

where $|O|^2 = OO^\dagger$

- The Bogomol'nyi bound and BPS equations for this system can be found using standard methods
  The total energy is given by

$$E = \int dt 2\pi\theta \mathrm{Tr}\mathrm{tr} \left[D_0 \phi (D_0 \phi)^\dagger + D_i \phi (D_i \phi)^\dagger + V(|\phi|^2) \right]$$

This can be written equivalently as

$$E = \int dt 2\pi\theta \mathrm{Trtr} \left[\left| D_0\phi \pm \frac{i}{\kappa}\left(v^2 - |\phi|^2\right)\phi \right|^2 + \left|\left(D_1 \pm iD_2\right)\phi\right|^2 \right.$$

$$\left. \pm \left(|\phi|^2 - v^2\right)\left(B - \frac{1}{\kappa}j_0\right) \mp \frac{1}{2}\epsilon_{kl}D_k j_l \pm v^2 B \right]$$

with $j_\mu = i\left[D_\mu\phi\phi^\dagger - \phi(D_\mu\phi)^\dagger\right]$ the current density associated with $U(1)_\star$ rotations and $B = F_{12}$ the magnetic field

The Bogomoln'yi bound is $E \geq v^2\,|\Phi|$ with $\Phi = \theta \mathrm{Trtr} B$ the total flux density

The bound is saturated for field configurations satisfying the following BPS equations:

$$(D_1 \pm iD_2)\,\phi = 0$$

$$D_0\phi \pm \frac{i}{\kappa}\left(v^2 - |\phi|^2\right)\phi = 0$$

Using the Gauss law constraint $\kappa B = j_0$
the second equation can be written equivalently as

$$B = \pm\frac{2}{\kappa^2}|\phi|^2\left(v^2 - |\phi|^2\right)$$

- The above equations can be solved exactly for nontopological vortex soliton solutions using the Harvey-Kraus-Larsen technique, which involves standard operators known as partial isometries

- In the Fock basis, one defines the projection and shift operators

$$S_m = \sum_{n=0}^{\infty} |n+m\rangle\langle n|$$

$$P_m = \sum_{n=0}^{m-1} |n\rangle\langle n|$$

and a further operator $Z_m = P_m a P_m + S_m a S_m^\dagger$

The shift operator is a non-unitary isometry:

$$S_m^\dagger S_m = 1$$

$$S_m S_m^\dagger = 1 - P_m$$

and the operator Z satisfies the following commutation relations:

$$[Z_m, Z_m^\dagger] = 1 - m|m-1\rangle\langle m-1|$$

- Now consider the covariant position operators of Gross, Hashimoto & Itzhaki:

$$X_i = x_i - \theta\epsilon_{ij}A_j$$
$$\tilde{X}_i = x_i - \theta\epsilon_{ij}\tilde{A}_j$$

The X can be used to measure the invariant size of the solutions: for any density associated with the solution (e.g. of energy or charge), a possible definition is

$$\Delta \equiv \sqrt{\frac{\mathrm{Trtr}\,(X_i - R_i)\,(X_i - R_i)}{\mathrm{Trtr}\rho}}$$

where $R_i \equiv \frac{\mathrm{Trtr}X_i\rho}{\mathrm{Trtr}\rho}$ denotes the density center

- The ingredients introduced above were used by Bak et al. to construct NC U(1) vortex solitons: defining

$$K \equiv \frac{1}{\sqrt{2\theta}}\,(X_1 - iX_2)$$

$$\tilde{K} \equiv \frac{1}{\sqrt{2\theta}}\left(\tilde{X}_1 - i\tilde{X}_2\right)$$

one finds that $B = \frac{1}{\theta}\left(i + \frac{1}{\theta}\,[X_1, X_2]\right)$ and that

the BPS equations can be written equivalently as

$$(K)^\dagger \phi - \phi a^\dagger = 0$$

$$1 - \left[K, K^\dagger\right] = \pm\frac{2\theta}{\kappa^2}|\phi|^2(v^2 - |\phi|^2)$$

- Bak et al. show that, for each value of B, there are two solutions of the type

$$\phi = \lambda_\pm |m-1\rangle\langle 0|$$
$$K = Z_m$$
$$\lambda_\pm^2 = v^2 \left[1 \pm \left(1 - \frac{2m\kappa}{\theta v^4} \right)^{\frac{1}{2}} \right]$$

The solutions exist provided $\theta \geq \frac{2m\kappa}{v^4}$ and in particular they admit no commutative limits as $\theta \to 0$

Once the BPS equations are solved, explicit operator expressions for the gauge field components and for Z can be given, viz.

$$A_0 = \mp \frac{1}{|\kappa|}\left(v^2 - |\phi|^2\right)$$

$$A_1 = \frac{i}{\sqrt{2\theta}}\left[P_m\left(a + a^\dagger\right)P_m + S_m\left(a + a^\dagger\right)S_m^\dagger - \left(a + a^\dagger\right)\right]$$

$$A_2 = \frac{1}{\sqrt{2\theta}}\left[P_m\left(a - a^\dagger\right)P_m + S_m\left(a - a^\dagger\right)S_m^\dagger - \left(a - a^\dagger\right)\right]$$

$$Z_m = \sum_{n=0}^{m-2}\sqrt{n+1}|n\rangle\langle n+1| + \sum_{n=0}^{\infty}\sqrt{n+1}|n+m\rangle\langle n+m+1|$$

Moreover it is straightforward to calculate the total flux, the energy and its distribution (according to the definition given earlier), the charge and the angular momentum carried by the solutions, e.g. one finds

$$\Phi = m$$
$$E = mv^2$$
$$Q = 2\pi\theta \mathrm{Tr}\mathrm{tr} j_0 = 2\pi\kappa m$$

- As we saw earlier, given a field configuration which solves the field equations of the NC Chern-Simons-Higgs model with gauge group $U(1,1)_\star \times U(1,1)_\star$ one can regard it as a solution of the theory of NC gravity with torsion (coupled to abelian CS gauge fields) studied by Cacciatori et al.
Explicitly, starting from the solutions of Bak et al. just described, the simplest way to construct a new gravitational solutions is to pick a $U(1)_\star$ suitably embedded in $U(1,1)_\star \times \mathrm{Id}$

- For any such embedding, the dreibein and connection can in principle be computed from the relations

$$\omega = \frac{1}{2}\left(A + \tilde{A}\right)$$
$$e = \frac{l}{2}\left(A - \tilde{A}\right)$$

The NC metric and torsion can then be found from the definition

$$G_{\mu\nu} \equiv e^a_\mu \star e^b_\nu \eta_{ab} \equiv g_{\mu\nu} + i b_{\mu\nu}$$

which implies

$$g_{\mu\nu} = \frac{1}{2}\left\{e^a_\mu, e^b_\nu\right\}_\star \eta_{ab}$$
$$b_{\mu\nu} = -\frac{i}{2}\left[e^a_\mu, e^b_\nu\right]_\star \eta_{ab}$$

- The calculations are algebraically quite tedious and it is difficult to find metrics in closed forms, however one can work perturbatively in the inverse of the noncommutativity parameter and at any rate the construction guarantees that the solutions thus constructed are exact

- Moreover the solutions
  - are localised in a region of the Moyal-Weyl plane of size $\simeq \sqrt{\theta}$
  - carry finite energy, charge and angular momentum

- The question of their spacetime interpretation is open...